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ELECTRICAL ENGINEERING

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BY

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PROPERTY OF ELECTRICAL LABORATORY,
FACULTY OF APPLIED SCIENCE.
Date Dec 30/25

WITH NUMEROUS ILLUSTRATIONS

METHUEN & CO. LTD.
36 ESSEX STREET W.C.
LONDON

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First Published in 1921

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PREFACE

THIS book is intended to give a survey of the principles of electrical engineering which shall be as complete as is practicable in one volume of moderate size. Commencing with the foundation experiments the aim has been to give in logical sequence the steps by which the practical formulæ and other results are reached.

The science of electrical engineering is based on fundamental experimental facts, which may be summarised as follows:

1. Coulomb's "inverse square" laws for electric charges and for magnetic poles.
2. Faraday's law of electromagnetic induction.
3. Ohm's law of electric resistance.
4. Faraday's laws of electrolysis.
5. Ampère's laws of the equivalence of electric currents and magnetic shells.
6. Joule's law of the heat energy of an electric current.

The plan of this book is formed on the recognition of this foundation basis of the science, and special emphasis has been placed on the importance of these laws.

In Section I. the principles of static electricity are dealt with in considerable detail.

This subject usually has been treated in a rather cursory manner in electrical engineering books. In view of the increasing importance to engineers of a thorough study of this branch, especially in connection with more recent developments, such as high-tension long-distance transmission, graded cables, wireless telegraphy, etc., the amount of space devoted to it in the book is thought to be justified. Generally, electrical engineering students are expected to obtain detailed knowledge of static electricity from books on physics, and as the writers of such books do not usually write from the engineer's point of view, their methods are not always so convincing to him as they might be.

Special attention has been given to the properties of magnetic shells and to Ampère's theorems of the equivalence of electric circuits and magnetic shells, since these theorems form the connecting link between the phenomena of magnetism and the magnetic effects of electric currents.

The data and tables of magnetic and electrical properties of materials and the chemical constants have been selected and arranged with great care to ensure that the information shall be as recent as possible. This is specially the case with regard to the magnetisation curves in Chapter VIII., and the data for iron alloys, iron losses, cobalt steel, etc. The writer is indebted to Messrs. Callender's Cable Co., Ltd., for the information on vulcanised bitumen given on p. 216. There do not seem to be any similar published data for this material, and the results given on p. 216 were obtained by Messrs. Callender when the book was in the press.

In Chapter XXI. the subject of transient currents is dealt with, and it is certain that in view of recent and prospective developments, transient phenomena will require very careful study by the practical scientific engineer.

In Chapter XXII. the method of dealing with alternating current problems by means of complex quantities is given at some length. This method forms a most powerful and simple means for attacking complicated problems, and its importance does not appear to be fully realised even yet by the majority of electrical engineers.

In Chapter XXIII. the relationships between electrostatic, electromagnetic, and practical units are explained, and throughout the book care has been taken to define clearly the units in which the results which have been obtained are expressed.

In general, the symbols recommended by the International Electrotechnical Commission have been used, but in all cases the meaning of each symbol used is explained in the text.

It is thought that no apology is necessary at this date for the use of the elementary calculus in a book of this kind.

No attempt has been made to give historical details as to the sources responsible for methods and results with the exception of a few cases, some of which have proved to be of classical importance.

The book is intended for students in universities and the higher classes of technical colleges, and it is hoped that electrical engineers in practice may find it useful as a book of reference.

The proofs have been examined with great care, and it is hoped that no error has remained undetected. Information as to any slips which may have crept in will be greatly appreciated by the writer.

Thanks are due to the various firms who have been good enough to lend blocks and to give information as to certain special apparatus.

Messrs. Methuen have spared no efforts to make the book as convenient, clear, and well arranged as possible, and the author has pleasure in acknowledging how much is due to their ready co-operation.

T. F. W.

SHEFFIELD,
December, 1920.

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ELECTRICAL ENGINEERING

SECTION I

STATIC ELECTRICITY

CHAPTER I

FIRST PRINCIPLES

1. Electricity developed by Friction.—If a rod of dry glass be rubbed with a piece of dry silk cloth, it is found that the two bodies adhere together, and some force is necessary to separate them, also when separated they attract each other if not too far apart. Further, if either of the rubbed surfaces be brought near light bodies, such as bits of paper, feathers, etc., these light bodies are attracted to the rubbed surface and adhere to it. The process of rubbing together has thus endowed both the glass and the silk with a new property—each is said to be **electrified.**

If a pith ball be suspended by a fibre of dry silk and allowed to touch the rubbed surface of the glass, it will be repelled after contact (Fig. 1). If a second pith ball, similarly suspended, be allowed to touch the rubbed surface of the silk cloth, it will also be repelled by the silk after contact. If, however, the two suspended pith balls be brought near together they will attract each other—in fact, each pith ball behaves to the other as the two rubbed surfaces of glass and silk behave to each other. Each pith ball has become electrified by contact with the rubbed surface.

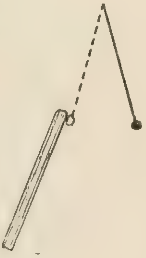

Fig. 1.

These experiments suggest that by friction between dry glass and dry silk something is developed which can be transferred to other bodies by contact. This something is called **electricity.**

A further inference from these experiments is that electricity of two kinds is developed, one kind appearing on the rubbed surface of the glass, and the other kind appearing on the rubbed surface of the silk. Two bodies similarly electrified repel each other—*e.g.*, the rubbed glass and the suspended pith ball after the two have been brought into contact with each other. Two bodies oppositely electrified attract one another—*e.g.*, the two suspended pith balls after they have been brought into contact with the rubbed glass and the rubbed silk respectively.

The kind of electricity developed on the surface of the rubbed glass is called **positive electricity,** and the kind of electricity developed on the surface of the rubbed silk is called **negative electricity.**

Glass and silk are not the only substances on the surfaces of which electricity develops by mutual friction. It is found that electricity is developed by friction on all bodies, and that substances may be classified in a list such that when any two are rubbed together the one which stands first in the list develops positive electricity, and the one that stands later in the list develops negative electricity. Such a list is as follows:

Fur.
Polished glass.
Flannel.
Pure cotton-wool.
Silk.
Shellac.
Ground glass.
Metals.

If, after rubbing the glass and silk together, the rubbed surfaces are left in contact and brought near an electrified pith ball, it is found that no evidence of electrification is exhibited by the adjacent surfaces of the glass and silk. As soon, however, as the two rubbed surfaces are moved apart, each separately shows evidence of electrification by its action on the pith ball. This experiment is one example of numerous others which confirm the following statement:

> **When electricity of one kind, say positive, is produced on one surface, a precisely equal amount of the opposite kind (negative) is produced on some other surface, so that if the two kinds are brought into contact, the amount of each is such as to exactly neutralise the effect of the other.**

Further examples of this fact will appear in following pages. See also Section III., § 133.

2. Conductors and Insulators.—If a metal ball be supported by a dry glass rod it may be electrified by contact with the rubbed surface of a piece of glass or silk, and its electrification may be proved by the fact that a pith ball suspended by dry silk fibre is repelled by the metal surface after being first brought into contact with it (Fig. 2). Moreover, the whole of the metal surface is found to be electrified. Some bodies, such as metals, graphite, and ordinary undistilled water, when electrified by contact at one point, are immediately found to be electrified over the whole surface. Such bodies are said to be good *conductors* of electricity. Many substances, however, such as glass, silk, shellac, india-rubber, mica, distilled water, when electrified at one part, only very gradually become electrified at other parts. Such bodies are said to be non-conductors or *insulators* of electricity.

3. Electrification by Induction.—When the rubbed glass and silk surfaces are separated, as has been said, equal amounts of positive and negative electricity appear on the respective surfaces. If now the silk is moved well away from the glass rod, the charge on the latter develops charges of negative electricity on the neighbouring bodies, such as the walls of the room, the table, etc. These neighbouring bodies are then said to have become charged by *induction*, or to have developed induced charges of electricity. Similarly, the silk develops charges of positive electricity on bodies in its neighbourhood, and if eventually it is brought into contact with a large conductor of electricity, such as the earth, the charge on the silk surface and the induced charge on the earth's surface combine and neutralise each other, and the silk is said to have become discharged.

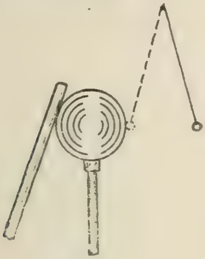

Fig. 2.

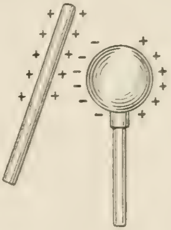

Fig. 3.

If now a metal ball is insulated by means of a glass-rod support, and is brought near to, but not touching, the charged glass rod, it is found that on the part of the surface of the ball near to the rod a negative charge appears, whilst on that part of the surface of the ball farther away from the glass rod a positive charge appears (Fig. 3).

If the glass rod be now moved away, the metal ball shows no signs of electrification.

It thus appears that a conductor which is not electrified contains equal quantities of positive and negative electricity which neutralise each other's effect.

The same result holds for non-conducting substances. In this case, however, the electricity cannot move freely over the surface as in conductors, and. therefore, if a charged rod be brought near an insulating substance, only that part of the insulator facing the rod will show signs of electrification.

When the metal ball is brought near an electrified body, the charge on the latter separates out the positive and negative electricity so that each appears at different parts of the surface.

Again, suppose that when the metal ball is brought near the charged glass rod it is connected to earth (Fig. 4) for a moment, and the earth connection then broken. It will then be found that when the rod is removed the ball remains charged with negative electricity (Fig. 5). The reason is that when the ball is connected to earth in the presence of the charged glass rod the positive electricity on the metal ball becomes neutralised by an equal charge of negative electricity from the earth connection, whereas the negative electricity on the ball remains bound by the positive charge on the glass rod. When the earth connection is broken and the glass rod removed, the metal ball remains charged with negative electricity, and is said to have been charged by induction. It thus appears that when charged by induction a body develops a charge of electricity of the opposite kind to that which it assumes when charged by contact with an electrified body.

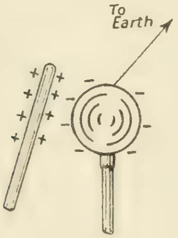

Fig. 4.

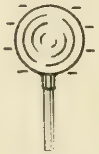

Fig. 5.

4. The Electric Field.—As has been seen, the neighbourhood of a charged body exhibits a special property—viz., if another charged body be brought to that neighbourhood it is acted on by a force, and if the body so introduced is light and free enough it will be set in motion. The neighbourhood of a charged body where this effect may be observed is called the **electric field,** due to the charge on the body. *An electric field is always bounded by surfaces charged with equal amounts of opposite kinds of electricity,* and may possess definite geometrical boundaries—*e.g.*, a charged body in a room—in which case the charged body and the walls, etc., of the room are the respective boundaries, or the field may extend indefinitely—*e.g.*, a charged body suspended in space and at an indefinitely great distance from any other body.

The insulating material in which an electric field exists is called the **dielectric.**

5. The Electric Charge resides on the Surface of a Conductor.—Consider an electrified metal ball A (Fig. 6) suspended by an insulating thread, and suppose BB are two hollow metal hemispheres provided each with an insulating handle, and such that when brought together to form a sphere, the internal diameter is somewhat larger than the diameter of the metal ball A.

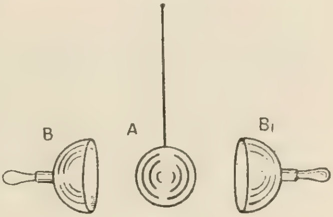

Fig. 6.

Now enclose the electrified ball A within the two metal hemispheres without allowing them to touch the ball, and then slightly raise or lower the hemispheres so that the internal surfaces touch the surface of the ball for a moment. If the hemispheres be now separated and removed from A, it is found on testing the ball A that it is completely discharged, whilst the hemispherical cups BB are found to be charged with electricity of the same kind as that with which the metal ball A was originally charged. This experiment is due to Cavendish, and shows that the electric charge resides on the surface of a conductor. Thus, as soon as the charged ball A touches the metal cups BB, the combination forms one conductor, and the charge passes to the surface of this conductor—viz., the outside of the metal cups BB.

A useful instrument for detecting small electrical effects is the gold leaf electroscope. Essentially this consists of a metal stem supported by the neck of a glass jar which serves to insulate the metal stem, and also to protect the delicate gold leaves from injury by draughts, etc. To the lower end of the metal stem two strips of gold leaf are secured, so that they hang free and parallel. To the top of the metal stem a metallic disc may be secured, which serves as a support for bodies whose electrical state it is desired to examine (Fig. 7). Various other refinements are used to increase the sensibility, reliability, and protection from damage, but the essential features are as described.

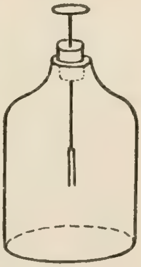

Fig. 7.

If a charged body be brought near the metal cap the leaves will diverge—each leaf being charged by induction and repelling the other.

Suppose a small metal pot be placed on the cap of the electroscope—the electroscope and pot being discharged. If, say, a positively charged metal ball, suspended by an insulating thread (*e.g.*, a silk fibre), be brought near the pot, the leaves of the electroscope will diverge. The divergence will increase as the charged ball is brought near the opening of the pot, and if the ball be gradually lowered inside the pot it will be observed that the divergence of the gold leaves continues to increase until the ball is well inside, but after that no difference is observed as regards the divergence of the gold leaves however the ball is moved about inside the pot (Fig. 8). Next, let the ball touch the inside of the pot; still the divergence of the gold leaves remains unchanged. If the ball is now removed, however, it is found to be completely discharged, and the leaves of the electroscope show no change when the ball is withdrawn, but maintain the same divergence. The process may be repeated, a fresh charge being given to the ball, and the ball lowered inside the pot and allowed to touch the inner surface. When the ball is withdrawn it is again found to be discharged. This experiment is one of a series made by Faraday, which established the important fact that the electric charge of a conductor resides on the outside surface.

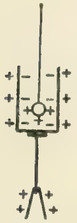

Fig. 8.

An interesting practical application was made by Clerk Maxwell of the fact that the charge resides on the outside surface of a conductor. When supporting a conductor, say a metal sphere, by means of an insulating handle, the charge gradually leaks along the handle if the latter is secured to the outside surface, and the conductor gradually becomes discharged. If, however, the ball is made hollow, and a hole made in the surface, the insulating support may pass clear through this hole and be secured to the inner surface of the conductor. Since there will be no charge on the inner surface and the supporting handle is clear of the outer surface, there is no tendency for the charge of the conductor to leak along this supporting handle.

An important practical application of the experiment illustrated by Fig. 8 is found in the induction electrical machine (see § 63).

6. Quantity of Electricity.—The experiments described in the preceding paragraph lead to the idea of a quantity of electricity, and show that electricity may be passed from one body to another as though it were a material substance. It may be said that an insulated charged conductor A is charged with the same quantity of electricity as the insulated charged conductor B, if the divergence of the gold leaves of the electroscope is the same when A and B are separately introduced inside, but not touching the pot placed on the cap of the electroscope. Further, the insulated conductor A is charged with a quantity of electricity equal to the sum of the quantities on the insulated conductors B and C, if the divergence of the gold leaves when A is held inside the pot is the same as the divergence when B and C are held inside the pot simultaneously.

Further, consider an insulated charged metal ball A, and suppose a precisely similar uncharged insulated metal ball B is allowed to touch A for a moment and is then removed. By lowering first A and then B inside the pot without allowing either A or B to touch the pot, and by noting the divergence of the gold leaves, it will be found that the original charge on A has divided, so that an exactly equal amount (*i.e.*, half the original charge on A) is now on each ball A and B. This division is just what would have been expected from considerations of symmetry. Since the two balls A and B are exactly alike, there is no reason why one ball should take more of the charge than the other when the two are brought in contact.

If a glass rod and piece of silk be rubbed together, and together inserted in the pot on the cap of the electroscope, no divergence of the leaves is observed. If, however, either the rubbed glass or the rubbed silk is separately inserted, the leaves will diverge. This experiment is a further proof of the statement made in § 1 that both kinds of electricity are developed simultaneously and in equal quantities.

Now consider the case of a charged metal ball suspended by an insulating thread of silk. If the charged ball be held inside the pot placed on the cap of the electroscope, the whole of its charge will correspond to an equal and opposite charge on the inside of the pot, and an equal charge of the same kind as that on the ball passes to the outside of the pot and the stem and leaves of the electroscope, so that the conditions are as shown in Fig. 8. If the charged ball is allowed to touch the inside of the pot its charge becomes neutralised by the charge on the inside of the pot, and the charge outside the pot remains unaffected. This experiment is often somewhat loosely explained by saying that the charge on the conductor introduced inside the pot passes to the outside of the pot when the charged conductor touches the inside of the pot.

7. The Mutual Action between two Small Electrified Bodies.—The Law of Inverse Squares.—It has already been seen that if two pith balls be suspended each by a silk thread and then brought separately into contact with a rubbed glass rod, the two balls repel each other when near together. It has also been seen that this experiment illustrates a fundamental property of electricity—viz., that like charges repel each other.

In order to investigate this property quantitatively, Coulomb devised the Torsian Balance. This instrument is shown in Fig. 9, and consists essentially of a thin rod of glass or shellac, *b d*, suspended horizontally by means of a very fine vertical wire. The rod carries a small ball b of gilt pith at one end, and a second similar ball a, also carried by a glass rod, is supported in a fixed position on the circumference of the circle in which the ball b moves when the suspension wire is twisted. The upper end of the fine suspension wire is fixed to a disc called the *torsion head*, which can be turned about a vertical axis and its position determined by means of a graduated horizontal scale. Fixed to the glass case of the instrument is another scale, by means of which the relative position of the fixed ball a and the movable ball b may be indicated. When the wire is without twist and the balls a and b without charge the torsion head to which the wire is secured points to the zero of its scale; also the ball b touches the ball a, and would occupy the position of a if the latter were removed. The rod bd is kept horizontal by means of small weights fixed at c and d.

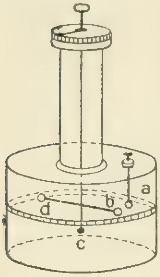

Fig. 9.

Now suppose each ball be electrified with, say, positive electricity, the ball b will consequently move away from the fixed ball a. Suppose the ball b moves through an angle δ. It will be possible to bring the ball b back towards the ball a by twisting the torsion head to which the upper end of the wire is secured. Suppose that β is the angle through which the torsion head must be turned in order to make the angular displacement of the ball b relatively to a equal to α. The total twist of the wire will then be $\alpha+\beta$. The twisting moment exerted on the wire by the repulsion between the charges on the ball a and b is equal to $\boldsymbol{F\,l}$ very approximately, for small values of α, where $2l$ is the length of rod bd.

δ

The opposing turning moment of the twisted wire is $C(\alpha+\beta)$ where C is a constant of the wire.

Hence $$F\,l=C(\alpha+\beta).$$

The distance between the balls a and b, also for small values of the angle α, is very approximately

$$x=l\alpha.$$

When a number of readings were taken in which the values of the angle α were $\alpha_1:\alpha_2:\alpha_3\ldots$ and the corresponding values of β were $\beta_1:\beta_2:\beta_3\ldots$ it was found that

$$\alpha_1^2(\alpha_1+\beta_1)=\alpha_2^2(\alpha_2+\beta_2)=\alpha_3^2(\alpha_3+\beta_3)\ldots=\text{a constant}:$$

that is— $$\frac{F_1}{F_2}=\frac{\alpha_2^2}{\alpha_1^2}=\frac{x_2^2}{x_1^2},$$

or $$F_1x_1^2=F_2x_2^2=\ \ldots\ =\text{a constant}.$$

This is the "*Law of Inverse Squares*," and applies equally to the attraction between oppositely charged bodies and the repulsion between similarly charged bodies. It is found to be more and more accurately fulfilled the smaller the bodies on which the charges are concentrated provided the force is great enough to permit of accurate measurement.

A more exact statement of the expression Fx^2 is as follows:

The turning moment on the rod *bd* is (see Fig. 10)—

$$F \cos \frac{\alpha}{2} l.$$

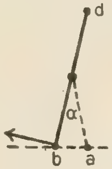

FIG. 10.

Hence $$F \cos \frac{\alpha}{2} l = C(\alpha + \beta).$$

This distance between the balls *a* and *b* is—

$$x = 2l \sin \frac{\alpha}{2}.$$

Hence $$Fx^2 = 4lC(\alpha + \beta) \sin \frac{\alpha}{2} \tan \frac{\alpha}{2}$$

$$= lC(\alpha + \beta)\, \alpha^2 \text{ approximately, as before.}$$

For values of α not greater than 30° the error in assuming that

$$\alpha^2 = 4 \sin \frac{\alpha}{2} \tan \frac{\alpha}{2}$$

is of the same order as the probable experimental errors of observation.

Now suppose that while the distance between the balls *a* and *b* is maintained constant by suitably adjusting the torsion head, the charge on one ball, say the movable ball *b* is constant, and the charge on the fixed ball is varied. Suppose that the twist on the wire necessary to keep the balls a certain distance apart is γ when each ball is charged. Now let the fixed ball be touched for a moment by a precisely similar insulated but uncharged ball and then the latter removed. The charge on the fixed ball *a* will thus be halved.

In order to maintain the distance between the fixed and movable ball the same as before, it will now be necessary to reduce the twist on the wire to $\frac{\gamma}{2}$. That is to say, the force between the two charged balls is now halved.

These results may be generalised as follows:

> **The electric force between two small electrified bodies is inversely as the square of the distance between them.**
>
> **The electric force between two small electrified bodies is directly proportional to the product of the two charges.**

If the charge on each body is of the same kind of electricity, the force between them is one of repulsion. If the charges on the two bodies are of opposite kinds the force is one of attraction.

Expressed in symbols these results may be written as follows:

$$F = \frac{q\,q_1}{\epsilon\,r^2},$$

where F is the force of attraction or repulsion between the two smaller bodies on which the charges are q and q_1 respectively, and r is the distance between them. The factor ϵ is a constant for any given medium in which the charged bodies are placed, but the value of ϵ is different for different media. For air the value of ϵ is assumed to be unity. A more detailed examination of the importance of the factor ϵ will be given in Chapter IV.

The constant is written $\frac{1}{\epsilon}$ in the above formula because ϵ turns out to be the specific inductive capacity, and writing thus avoids the introduction of another constant (see §§ 23, 47, and 48).

For purposes of calculation, electricity may be treated as though it were an actual substance, each particle of which acts on every other particle according to Coulomb's Laws.

8. Unit of Electrical Quantity.—As has been stated in the previous paragraph, the constant ϵ is assumed to have the value unity when the medium through which the electric forces act is air, that is to say, the force between the two charged bodies in air is given by the expression—

$$F = \frac{q\,q_1}{r^2}.$$

Suppose now that the values of q and q_1 are equal, and are such that the force F is one dyne when the distance r is one centimetre. It follows that q and q_1 must each have the value unity in this system of units—*i.e.*, the system in which the constant ϵ is assumed to be unity, the unit of force is the dyne, and the units of length, mass, and time are respectively the centimetre, the gramme, and the second (see Section VI., § 289). The definition of the *centimetre-gramme-second electrostatic unit of electrical quantity* may therefore be stated as follows:

> **The c.g.s. electrostatic unit of electrical quantity is that quantity which acts with a force of one dyne on an equal quantity at a distance of one centimetre in air.**

In referring to unit quantity of electricity unit *positive* quantity is usually implied.

Some idea of the magnitude of this unit quantity may be obtained by noting that if two pith balls, each weighing 1 gm. and hung by silk fibres 1 metre long without appreciable weight, equally electrified so as to make them separate from each other 10 cms., the charge on each ball must be about 70 electrostatic c.g.s. units of quantity

The c.g.s. electrostatic unit of electrical quantity is very small in comparison with the quantities which usually come into account in practice, and consequently a multiple of this unit—viz., the **Coulomb**—is generally used. The Coulomb is 3×10^9 times as large as the c.g.s. electrostatic unit, and the reason why this particular number is chosen as the multiple will appear later (see Chapter XXIII.).

9. Electric Force or Electric Intensity.—The electric force or electric intensity at any point in an electric field is the force which would be experienced by unit positive charge when placed at that point *under the assumption that introducing the unit charge into the electric field does not modify the distribution of electricity on the charged bodies to which the electric field is due.*

Thus, if the field is relatively a weak field, the intensity at any point might be defined as 1,000 times the force experienced by $\frac{1}{1000}$ of a unit of positive electricity placed at that point, or more generally the electric force at any point in an electric field is x times the force experienced by $\frac{1}{x}$ unit of positive electricity placed at that point, the value of $\frac{1}{x}$ being sufficiently small to ensure that the distribution of electricity on the charged bodies to which the field is due is not appreciably altered by the introduction of $\frac{1}{x}$ the unit of electricity at the point in question.

The direction of the electric intensity at any point in an electric field is the direction in which a positively charged small body would tend to move if placed at that point.

The electric intensity due to a charge q electrostatic c.g.s. units at a distance r cms. from the charge will be (§ 8)—

$$\frac{q}{\epsilon r^2} \text{ dynes.}$$

The value of ϵ depends upon the nature of the insulating medium of the field of force (see § 48).

Since for air ϵ is assumed to be unity, the intensity of the force at a point in air due to a charge q at distance r is—

$$\frac{q}{r^2} \text{ dynes.}$$

10. Lines of Force.—Lines of electric force are lines drawn in the field, and such that the direction of the line through any point gives the direction of the force at that point.

11. Unit Line of Force.—Suppose a c.g.s. unit of electricity is placed at the centre of a spherical surface of 1 cm. radius. The force experienced by another unit charge, placed anywhere on the surface of the sphere, is, by definition, unity, and *one unit line of force is said to cross each square centimetre of the surface of the sphere.* The strength of an electric field may thus be represented by the number of unit lines of force which cross one square centimetre of area, the area being such that its plane is perpendicular to the direction of the lines of force at the place considered.

Since one unit line of force crosses each square centimetre of area of the surface of the sphere of unit radius at the centre of which a unit charge is placed, it follows that 4π lines cross the whole surface of the sphere, or, otherwise stated, each unit charge of electricity gives rise to 4π unit lines of force.

12. Flux of Force.—The flux of force across any area in an electric field is the number of unit lines of force which cross that area.

Another way of expressing the flux across an area is as follows:

Let F be the intensity of the force at a point in the area δS and let F_n be the resolved component in a direction perpendicular to the area δS, then the flux across the area δS is—

$$F_n \, \delta S.$$

From what has been said in § 11 it will be seen that the flux due to unit charge is 4π.

13. Tubes of Force.—Let A and B be two charged surfaces, forming the boundaries of an electric field (Fig. 11). On the surface A draw an element of area δS, and at every point of the contour of this element let lines of force be drawn. These lines will enclose a tube which will reach to the surface B. This tube is called a *tube of force.* If F is the intensity of the force at the point P in the element of surface the product F δS is the *flux of force* within the tube, assuming that the intensity of the force is uniform over the area δS.

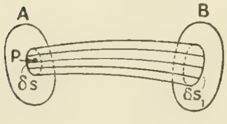

FIG. 11.

14. There is no Electric Force or Electric Charge within the Material of an Electrified Conductor.—This fact is proved by experimental evidence, such as is given by the experiments described in § 5. The following experiment made by Faraday provides further verification. A large chamber was constructed with walls of conducting material, and supported on insulating feet. The chamber was large enough for Faraday to get inside and experiment as to the electric charges inside. The outside was connected to an electrical machine, and so strongly electrified that large sparks could be drawn from the outside surface. Not the smallest trace of electricity or electric force could be detected inside, even by means of the most delicate instruments.

The following is an extract from Faraday's account of the experiment:

"A slight cubical wooden frame was constructed, and copper wire passed along and across it in various directions, so as to make the sides a large network, and then all was covered in with paper, placed in close connection with the wires, and supplied in every direction with bands of tinfoil, that the whole might be brought into good metallic communication and rendered a free conductor in every part. This chamber was insulated in the lecture-room of the Royal Institution. . . . I went into the cube and lived in it, and used lighted candles, electrometers, and all other tests of electrical states. I could not find the least influence upon them, or indication of anything particular given by them, though all the time the outside of the cube was powerfully charged, and large sparks and brushes were darting off from every part of its outer surface."

15. The Electric Force at the Surface of a Conductor is everywhere Normal to the Surface of the Conductor.—That this must be so follows from the consideration that if there were any tangential component of the electric force there would be a movement of electricity along the surface until the force became normal (see also §§ 33 and 62).

16. Electric Surface Density.—If a quantity of electricity of q electrostatic c.g.s. units exist on a surface of area S sq. cms., the average electric surface density σ is $\frac{q}{S}$, and the density at any given point on an electrified surface is given by—

$$\sigma = \frac{dq}{dS} \text{ electrostatic c.g.s. units per sq. cm.,}$$

where dq is the small element of charge on the extremely small element of surface δS containing the point considered.

From unit surface in air on which the density of the electric charge is $+\sigma$, $4\pi\sigma$ unit lines of force start, and at unit surface on which the density is $-\sigma$, $4\pi\sigma$ unit lines end (see § 23).

17. Distribution of Surface Density over the Surface of a Conductor.—Experiments conclusively prove that when a charge of electricity is given to an insulated conductor, the electricity resides on the outside surface, and that there is neither electric charge nor electric force within the conductor.

When a charge is given to a conductor, the electricity will therefore spread over the surface, so that the distribution will give no resultant force inside the conductor. It then becomes a mathematical problem to determine the surface density at every point on the surface of the conductor necessary to fulfil this requirement, and also to fulfil Coulomb's Law of the mutual force between elementary charges of electricity (see § 7).

It is only in a few special cases that this mathematical problem can be solved, but, generally speaking, the distribution will be such that the density will be greatest on sharp projections of the surface, such as projecting points and edges (see § 35, Fig. 38).

18. Representation of the Electric Field by Lines of Force.—Suppose an area δS is drawn at a place in the electric field, so that the plane of the area is perpendicular to the direction of the lines of force at that place. Let the intensity of the electric force over the area δS be F. From the definition of unit line of force it follows that there will be F unit lines of force crossing every square centimetre of the area δS—that is to say, the flux of force across the area δS will be F δS. Now draw the tube of force of which the area δS is a section, and let another right section of the tube be drawn, such as δS_1 (Fig. 12). Then it is clear that the flux of force is constant throughout the tube, assuming there are no free charges of electricity in the tube. Consequently the flux of force crossing the area δS must be the same as the flux which crosses the area δS_1. Hence, if F_1 is the electrical intensity over the area δS_1, the flux across the area δS_1 will be $F_1 \delta S_1$; therefore—

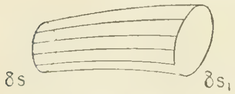

Fig. 12.

$$F\,\delta S = F_1\,\delta S_1.$$

It will be seen from this expression that if δS_1 is greater than δS the number of lines of force per unit of area at δS_1 will be less than the number of unit lines of force per unit of area at δS. Consequently, if a field of force be mapped out by drawing the unit lines of force in the field, the relative strength of the field at different places may be seen by the relative density of the lines of force at the places in question. Where the lines of force are relatively far apart, the field is relatively weak, and where the lines of force are close together the field is strong.

19. Method of drawing Lines of Force in some Typical Cases—*Example* 1. *An Electric Charge concentrated at an Isolated Point.*—Let the charge be q electrostatic c.g.s. units. The force at any point P will be in the direction of the line joining the point P to the charge, and by reasons of symmetry it is clear that the unit lines of force will be radial, uniformly divergent lines.

This result may also be arrived at as follows: The force at a point P distant r cms. from the charge will be the same at all points and equal to $\frac{q}{\epsilon r^2}$ dynes, that is to say, $\frac{q}{\epsilon r^2}$ lines will cross each square centimetre of area, the direction of the area being perpendicular to the line joining the point P and the charge q. Suppose now that a spherical surface be drawn with centre at the charge and of radius r cms. The number of lines of force which cross each square centimetre of area of the sphere will be $\frac{4\pi q}{\epsilon 4\pi r^2}$, *i.e.*, $\frac{q}{\epsilon r^2}$. Hence, if the lines of force be drawn radially uniformly outwards, they will satisfy the required conditions (see § 20).

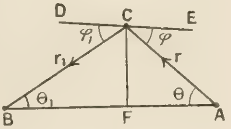

FIG. 13.

Example 2. Two Equal and Opposite Charges concentrated at Points A and B respectively (*Figs.* 13 *and* 14).—Suppose at point A one electrostatic c.g.s. unit of positive electricity is placed, and at the point B one electrostatic c.g.s. unit of negative electricity. Consider the force at a point C. Let the distance AC be r cms. and the distance BC be r_1 cms. Let the angle CAB be θ and the angle CBA be θ_1.

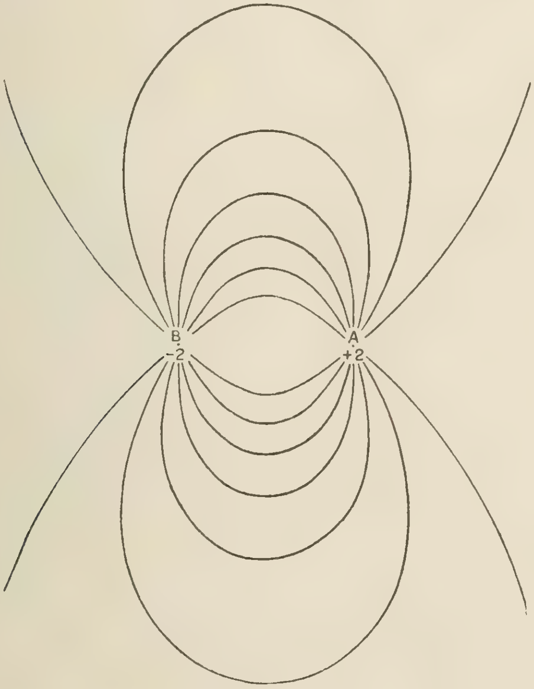

FIG. 14.

The force due to the charge at A will be $\frac{1}{\epsilon r^2}$ dynes, and will be directed away from A. The force at C due to the charge at B will be $\frac{1}{\epsilon r_1{}^2}$ dynes, and will act in the direction CB. Suppose DCE is the tangent to the line of force passing through C. By the definition of a line of force it follows that the resultant in a direction perpendicular to DCE will be zero.

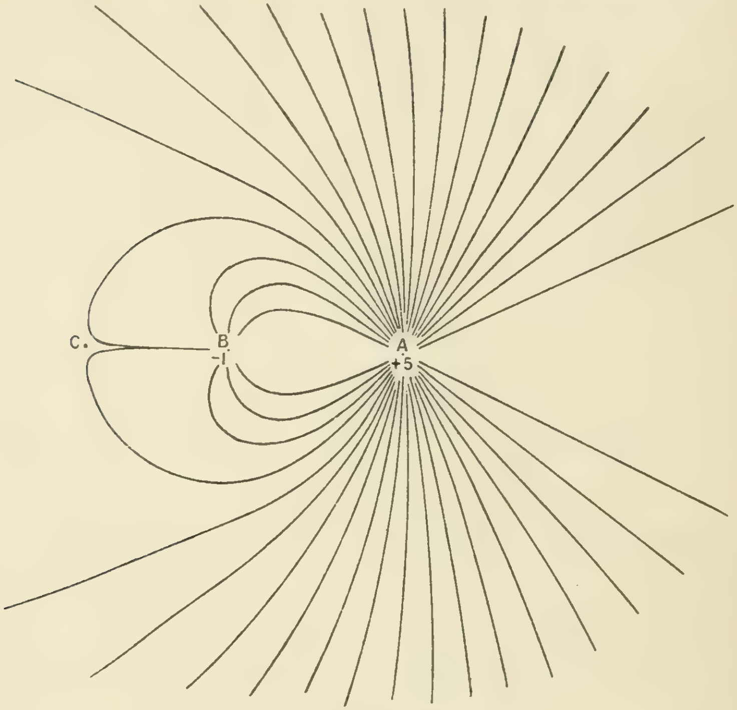

FIG. 15.

Let the angle ACE be ϕ and the angle BCD be ϕ_1.

Then $$\frac{1}{\epsilon r^2} \sin \phi = \frac{1}{\epsilon r_1{}^2} \sin \phi_1.$$

Further, if CF is drawn perpendicular to AB,

$$\mathrm{CF} = r \sin \theta = r_1 \sin \theta_1.$$

Also $r_1 \frac{d\theta_1}{ds} = \sin \phi_1$: $-r \frac{d\theta}{ds} = \sin \phi$: in which ds is a very small element of length of the line of force passing through C, the positive direction of the line being considered as from A towards B.

Hence $$\frac{1}{\epsilon r_1}\frac{d\theta_1}{ds}+\frac{1}{\epsilon r}\frac{d\theta}{ds}=0,$$

or $$\frac{\sin\theta_1}{\epsilon}\frac{d\theta_1}{ds}+\frac{\sin\theta}{\epsilon}\frac{d\theta}{ds}=0;$$

that is, $$\cos\theta_1+\cos\theta=\text{a constant.}$$

The lines of force shown in Fig. 14 have been drawn in this way, each line of force corresponding to a different value of the constant.

If instead of unit positive charge at A and unit negative charge at B there are q units of positive electricity at A, and q_1 units of negative electricity at B, the expression for the lines of force will be—

$$q_1\cos\theta_1+q\cos\theta=\text{a constant,}$$

and to each value of the constant a line of force corresponds.

In Fig. 15 the lines of force represent the case in which the charge at A is 5 units of positive electricity, and the charge at B is 1 unit of negative electricity—that is, the equation for the lines of force will be—

$$5\cos\theta+\cos\theta_1=\text{a constant.}$$

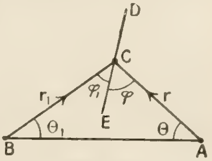

FIG. 16.

Example 3. *Like Charges concentrated at the Points A and B.*—In Fig. 16 let 1 unit of positive electricity be concentrated at the point A, and 1 unit of positive electricity be concentrated at B. Consider the forces at a point C due to the two charges. The force at C due to the charge at A will be $\frac{1}{\epsilon r^2}$ and will act in the direction AC. The force at C due to the charge at B will be $\frac{1}{\epsilon {r_1}^2}$, and will act in the direction BC. The resultant force in the direction perpendicular to the tangent to the line of force at C—viz., perpendicular to DCE—must be zero.

Therefore $$\frac{1}{\epsilon r^2}\sin\phi=\frac{1}{\epsilon {r_1}^2}\sin\phi_1.$$

But $$r\frac{d\theta_1}{ds}=\sin\phi_1 : r\frac{d\theta}{ds}=\sin\phi,$$

and $$r_1\sin\theta_1=r\sin\theta.$$

Hence $$\frac{1}{\epsilon r}\frac{d\theta}{ds}-\frac{1}{\epsilon r_1}\frac{d\theta_1}{ds}=0,$$

and $$\frac{\sin\theta}{\epsilon}\frac{d\theta}{ds}-\frac{\sin\theta_1}{\epsilon}\frac{d\theta_1}{ds}=0;$$

that is, $$\cos\theta-\cos\theta_1=\text{a constant.}$$

In Fig. 17 are shown the lines of force drawn in this way for the case in which 5 units of positive electricity are concentrated at A, and 1 unit of positive electricity is concentrated at B. It is to be observed that the force at the point C is zero, and the dotted curve passing through C divides the lines of force emanating from the charge at A from those emanating from the charge at B.

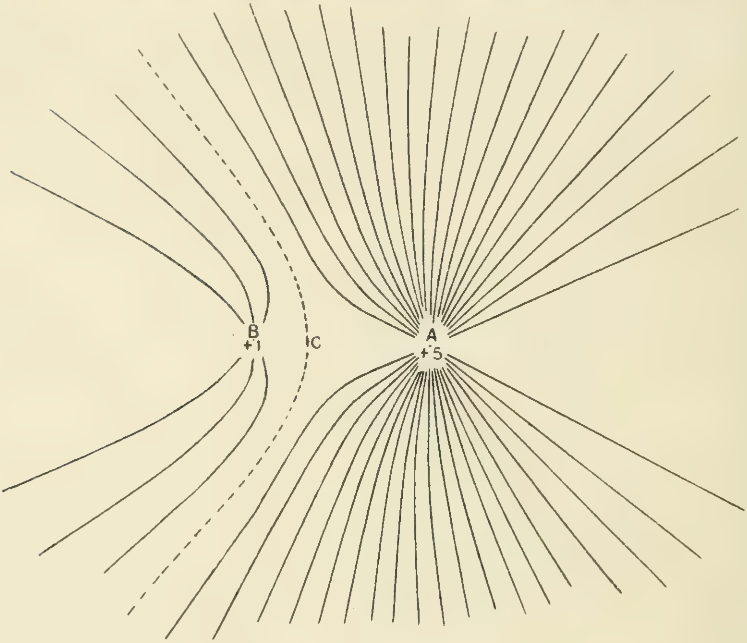

Fig. 17.

The equation for the lines of force shown in Fig. 17 is—

$$5 \cos \theta - \cos \theta_1 = \text{a constant.}$$

In Fig. 18 are shown the lines of force due to 2 units of positive electricity concentrated at the point A, and 2 units of positive electricity concentrated at B. Of course, the lines of force shown in Figs. 17 and 18 apply equally well if the charges at the points A and B are both negative.

In Figs. 14, 15, 17, and 18, if the system be revolved about the axis AB, consecutive lines of force will generate the boundaries of adjacent tubes of force, and in these figures the lines are so drawn that each of such tubes of force contain the same flux of force. That is to say, if a small sphere be drawn with centre at A or B each tube of force which meets the sphere will cut off the same area from the surface of the sphere.

In Fig. 14 a charge of + 2 units is shown at A and a charge of – 2 units at B. There are eight tubes of force leaving A and ending on B, and each tube contains $\frac{2 \times 4\pi}{8} = \pi$ units of flux (see § 11). In Fig. 15, twenty tubes of force start from the charge of + 5 units at A, and of these four tubes reach the charge of – 1 unit at B, and the remaining sixteen tubes from A go off to infinity. In this case also each tube contains π units of flux.

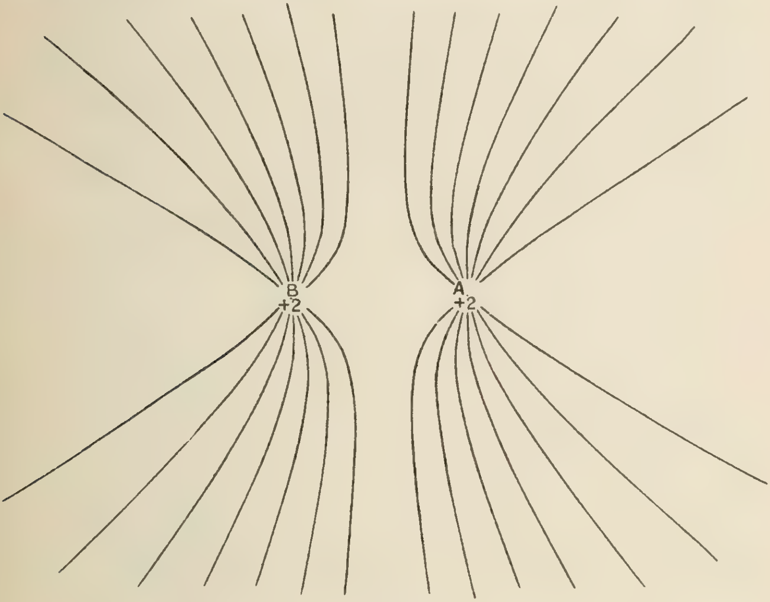

FIG. 18.

Similarly for the cases shown in Figs. 17 and 18.

In each of the Figs. 14, 15, 17, and 18 the distance A B is assumed to be 5 cms.

CHAPTER II

SOME DEDUCTIONS FROM COULOMB'S LAW OF THE FORCE BETWEEN ELECTRIFIED PARTICLES

In Chapter I. it has been shown how, by applying Coulomb's Law, the lines of force between two electrified particles may be drawn. In the present chapter some further important deductions from Coulomb's Law will be made.

20. Gauss' Theorem—Consider a Closed Surface drawn in any Way in an Electric Field—The Total Flux of Force across this Surface is equal to the Quantity of Electricity enclosed by the Surface multiplied by $\frac{4\pi}{\epsilon}$, the Flux in the Outward Direction being considered Positive, and the Flux in the Inward Direction being considered Negative.—As a preliminary step in the proof of this theorem it is to be observed that if a charge q units be placed at the point o, and if a cone of solid angle $\delta\omega$ be supposed to start from this point, the flux within the cone is constant.

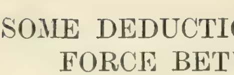

Fig. 19.

For consider two right sections of the cone—viz., δS_1 and δS_2 (Fig. 19)—at distances r_1 and r_2 cms. respectively from the apex of the cone. The value of the force at δS_1 is $\frac{q}{\epsilon r_1^2}$.

The flux across δS_1 is—

$$\left[\frac{q}{\epsilon r_1^2}\right] r_1^2\delta\omega = \frac{q}{\epsilon}\delta\omega,$$

since the area δS_1 is $r_1^2\delta\omega$ sq. cms.

Similarly the flux across δS_2 is—

$$\left[\frac{q}{\epsilon r_2^2}\right] r_2^2\delta\omega = \frac{q}{\epsilon}\delta\omega.$$

Hence the flux across any section of the cone is constant

This result also follows from the deduction in § 19, Example I, that the lines of force due to a charged particle are straight lines radiating uniformly from the charge.

Now consider an imaginary closed surface (*i.e.*, not a conducting surface), drawn in the electric field, and suppose the electric charge q is placed at a point V outside the surface (Fig. 20). Draw a cone of indefinitely small angle $\delta\omega$, and let it intercept the surface at δS_1 and δS_2. The flux $\frac{q\delta\omega}{\epsilon}$ is the same for each element of surface intercepted by this cone.

The flux, however, enters the closed surface at δS_1, and leaves the closed surface at δS_2, and consequently the total flux entering or leaving the closed surface through the cone is zero. Since the whole surface may be divided up into pairs in this way by means of elementary cones, and since the total flux entering or leaving the surface through each cone is zero, it follows that when the charge q is outside the closed surface the net total flux entering or leaving the surface is zero.

Next suppose that the charge q is at a point V inside the closed surface (Fig 21).

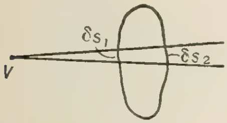

Fig. 20.

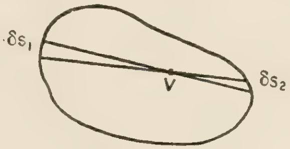

Fig. 21.

Again, draw a cone of indefinitely small angle with apex at V, and let it intercept on the closed surface the elements of area δS_1 and δS_2. Then the flux across the element δS_1 is $\frac{q\delta\omega}{\epsilon}$, and the flux across the element δS_2 is also $\frac{q\delta\omega}{\epsilon}$. Moreover, in this case the flux for each of the elements δS_1 and δS_2 is in the direction of leaving the surface if the charge at q is positive, or entering the surface if the charge at q is negative. Hence the total flux entering or leaving the surface due to the charge q at V inside the surface is the sum of the values $\frac{q\delta\omega}{\epsilon}$ taken over the whole surface—*i.e.*, is equal to—

$$\frac{4\pi q}{\epsilon}$$

If there be a number of charges q_1, q_2, q_3 . . . inside the surface, it is clear that the total flux through the surface will be $\frac{4\pi}{\epsilon}\,[\pm q_1 \pm q_2 \pm q_3 \pm \ . \ . \ .]$ the + or − sign being taken according as the respective charges are positive or negative.

21. The Electric Force at any Point outside a Charged Spherical Conductor is the same as if the Charge were concentrated at the Centre of the Sphere.—This may be proved very simply by the application of Gauss' theorem proved in the preceding paragraph. Let σ be the surface density of the charge on the sphere. It will be obvious that the surface density σ will be the same for the whole surface of the sphere, and the lines of force due to the charge on the sphere will radiate outwards uniformly. The total flux issuing from the charged surface of the sphere will be—

$$\frac{4\pi r^2\sigma\ 4\pi}{\epsilon},$$

where r is the radius of the sphere.

To find the intensity at a point P distant D cms. from the centre of the sphere, consider a spherical surface drawn concentric with the charged sphere and passing through the point P. The total flux passing through this spherical surface will be the same as that proceeding from the charged surface of the sphere—*i.e.*,

$$\frac{16\pi^2 r^2\sigma}{\epsilon}.$$

The intensity over this imaginary surface will be uniform, and hence the intensity at the point P will be—

$$\frac{16\pi^2 r^2\sigma}{\epsilon}\ \frac{1}{4\pi \mathrm{D}^2}$$

$$=\frac{4\pi r^2\sigma}{\epsilon \mathrm{D}^2}$$

$$=\frac{\mathrm{Q}}{\epsilon \mathrm{D}^2},$$

where Q is the total charge on the sphere.

Hence the intensity at the point P is the same as if the whole charge Q were concentrated at the centre of the sphere, and the lines of force due to the charged sphere will be straight uniformly radiating lines, as stated above (see also § 19, Example 1).

If the point P is close to the surface of the sphere—*i.e.*, if OP is very approximately equal to r the radius of the sphere—the force due to the charge of density σ in electrostatic c.g.s. units per sq. cm. on the surface of the sphere is—

$$\frac{4\pi\sigma}{\epsilon}\ \text{dynes};$$

and the force acts along the normal to the surface, outwards if Q is positive, and inwards if Q is negative. This result for the force close to the surface of a charged sphere is an example of the general case proved in § 23.

22. The Electrical Force inside a Charged Spherical Conductor is Zero.—Take any point P inside the charged sphere

(Fig. 22) and consider the double elementary cone of which P is the common apex. Let $\delta\omega$ be the solid angle at the apex of the cone. Let σ be the density of the charge on the surface of the sphere. From considerations of symmetry it will be clear that the surface density must be uniform over the whole surface of the sphere.

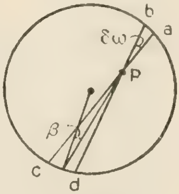

FIG. 22.

The surfaces of the sphere intercepted by the cones are shown at *ab* and *cd*. The projection of the surface *ab* in a direction perpendicular to the axis of the cone is *ab* cos β. Let r_1 be the mean distance of *ab* from the apex of the cone P. Then from the definition of a solid angle—

$$\frac{ab\cos\beta}{r_1^2}=\delta\omega.$$

Similarly for the surface *cd*. If r_2 is the mean distance of *cd* from the apex of the cone P, then—

$$\frac{cd\cos\beta}{r_2^2}=\delta\omega;$$

or

$$\frac{cd}{r_2^2}=\frac{ab}{r_1^2}$$

Now consider the force due to the charge on the portion *ab* of the surface of the sphere. Since by § 21 all the lines of force due to the charged sphere are directed through the centre of the sphere, the resultant force due to the charge on *ab* will also be directed through the centre of the sphere. The component of the force in the direction through P and at the distance of P from *ab* is—

$$\frac{\sigma\, ab\cos\beta}{\epsilon r_1^2}$$

Similarly for the area *cd* the force at P is—

$$\frac{\sigma\, cd\cos\beta}{\epsilon r_2^2}$$

But the direction of the force due to the area *cd* is opposite to that due to the area *ab*. Hence the total force at P due to the charged areas *ab* and *cd* is—

$$\frac{\sigma}{\epsilon}\left[\frac{ab}{r_1^2}-\frac{cd}{r_2^2}\right]\cos\beta.$$

This expression vanishes since the quantity in the brackets has already been shown to be zero. Now the whole surface may be divided up by means of double elementary cones, of which the common apex is P, and for each pair of corresponding elements of the surface of the sphere, such as the areas *ab* and *cd*, the resultant force at P vanishes. It therefore follows that the electric force inside a charged spherical conductor vanishes, and this result is true whatever the value of ϵ may be.

Since the result just arrived at is a consequence of the law of inverse squares, the experimental evidence that there is no electric force inside such a charged sphere is often given as a proof of this law, more rigorous than that deduced from the experiments with the Torsion Balance. As to objections to this, however, see Poynting (B. A. Report, 1886).

Cavendish used this principle to verify the law of inverse squares, and Clerk Maxwell repeated it, using more delicate instruments than those which were at the disposal of Cavendish.

23. The Electric Force at a Point indefinitely near the Surface of a Charged Conductor in Equilibrium, whatever be the Acting Charges, is equal to $\frac{4\pi\sigma}{\epsilon}$ where σ is the Electric Density near this Point (Coulomb's Theorem).—Take an element of surface δS sq. cm. on an electrified conductor in equilibrium in the presence of any electric charges (Fig. 23). Consider a tube of force of which δS is one end. If F is the force just outside the element δS and σ is the surface density of the charge on δS measured in electrostatic c.g.s. units per sq. cm., the flux of force in the tube is FδS. But by Gauss' theorem (§ 20), the flux due to the charge $\sigma\delta$S is—

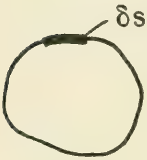

Fig. 23.

$$\frac{4\pi\sigma\delta S}{\epsilon}.$$

By § 14 there is no flux in the material of the conductor;

therefore $$F\delta S=\frac{4\pi\sigma\delta S}{\epsilon},$$

or $$F=\frac{4\pi\sigma}{\epsilon}\text{ dynes.}$$

The direction of this force is perpendicular to the surface of the conductor, as was explained in § 15.

24. The Electric Force inside a Homogeneously Charged Solid Sphere.—It may be observed that the only way in which such a charged sphere may be realised in practice is by building it up of a large number of concentric spherical shells insulated from each other.

Let Q electrostatic c.g.s. units be the total charge of the sphere, and let R cms. be the radius. At a point P inside the sphere and distant r cms. from the centre the force due to that part of the charge contained in a sphere of radius r is—

$$\frac{1}{\epsilon r^2}\frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3}Q$$

$$=\frac{r^3}{R^3}Q\frac{1}{\epsilon r^2}$$

$$=\frac{rQ}{\epsilon R^3}\text{ dynes.}$$

The force at P due to the part of the charge contained in the spherical shell of internal radius r and external radius R is zero, as has already been proved in § 22.

Hence the force at a point P in the sphere distant r cms. from the centre is—

$$\frac{rQ}{\epsilon R^3}\text{ dynes,}$$

or the force is directly proportional to the distance from the centre. This expression is similar to that for the force on a mass inside the earth.

25. The Electric Force due to a very Long Straight Wire at a Point distant D cms. from the Axis of the Wire where D is Small compared with the Length of the Wire.—Let q electrostatic c.g.s. units be the charge per cm. length of the wire. Consider an element of wire of length δl cm. (Fig. 24). The force at P due to the charge on δl will be $\frac{q\delta l}{\epsilon r^2}$ dynes in the direction of F.

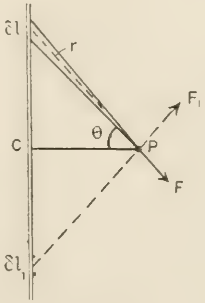

Fig. 24.

But

$$r\cos\theta=CP=D;\text{ and } r\frac{\delta\theta}{\delta l}=\cos\theta.$$

Hence

$$F=\frac{q\delta\theta}{\epsilon D}\text{ dynes.}$$

The component of F in the direction CP is—

$$F\cos\theta=\frac{q\cos\theta\delta\theta}{\epsilon D}\text{ dynes.}$$

The component of F in the direction perpendicular to CP will be neutralised by the component in the direction perpendicular to CP, and due to an element δl_1 symmetrical with δl about the point C. Hence the total force at P due to the charged wire will be in the direction CP and will be—

$$\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{q \cos\theta}{\epsilon D} d\theta$$

$$=\frac{2q}{\epsilon D} \text{ dynes.}$$

The limits of the integration are $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, since the length of the wire is assumed to be great as compared with the distance D. Hence the force at P due to the charged wire is inversely proportional to the distance D from the wire, at points for which the distance D is small compared with the length of the wire. The direction of the force is perpendicular to the wire.

Another method of arriving at this result is as follows:

In the first place, considerations of symmetry show that the intensity at points such as P is in a direction perpendicular to the axis of the wire, and is the same at all points on a circle of radius D with centre on the axis of the wire and the plane of the circle perpendicular to the axis of the wire.

Consider a slice of the wire bounded by the two parallel planes BE and CF close together and perpendicular to the axis of the wire (Fig. 25). Let x cms. be the small distance between the planes, and let q electrostatic c.g.s. units be the charge per cm. length of the wire. The charge on the slice of wire between the planes will be qx.

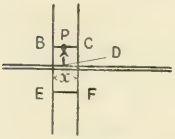

Fig. 25.

Now let a cylindrical surface be drawn coaxially with the wire, and passing through P, cutting the planes at BC and EF. From what has been said it will be seen that the direction of the flux due to the charge on the wire between the planes will be parallel to these planes, and therefore no flux will pass through the ends of the cylinder. By Gauss' theorem (§ 20), the total flux due to the charge on the element of wire between the planes is $\frac{4\pi qx}{\epsilon}$.

Hence the intensity at the point P distant D cms. from the axis of the wire will be—

$$\frac{4\pi qx}{\epsilon} \frac{1}{2\pi D x} \text{ dynes,}$$

that is,

$$\frac{2q}{\epsilon D} \text{ dynes.}$$

26. The Electric Force between two Oppositely Charged Discs arranged parallel to each other and at a Small Distance

apart.—Let AB and CD be the two charged discs arranged parallel to each other, and suppose that the distance apart d cms. is small compared with the diameter of each disc (Fig. 26). Suppose the inner face of the disc AB is charged with a uniform surface density $+\sigma$ and the inner face of the disc CD is charged with a uniform surface density $-\sigma$ electrostatic c.g.s. units per sq. cm. It may be noted in passing that when a charge is given to either of the plates AB or CD arranged parallel to one another and close together, it can be deduced from Coulomb's Law that, except at places near the edges of the discs, the charge will so distribute itself over the surface that there will be uniform density over the inner face of each plate.

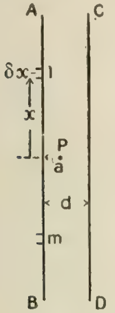

FIG. 26.

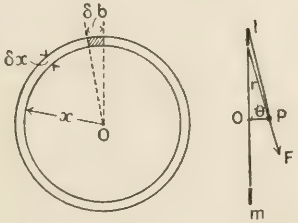

FIG. 27.

First consider the force at a point P in between the discs and lying on the common axis, the point being distant a cms. from the plate AB and distant $d-a$ cms. from the plate CD. Now draw a circular strip lm on the face of the plate AB of radius x cms. and breadth δx (see Figs. 26 and 27). An element of the surface of this strip of width δb will have a charge $\sigma\delta b\delta x$.

The force at P due to this charge will be—

$$F = \frac{\sigma\delta b\delta x}{\epsilon r^2} \text{ dynes.}$$

The component of F in the direction OP will be—

$$\frac{\sigma\delta b\delta x}{\epsilon r^2} \cos\theta \text{ dynes}$$

The component of F in the direction perpendicular to OP will clearly be neutralised by the component in the opposite direction, due to the diametrically opposite element to $\delta b\delta x$, so that for the whole circular strip the only effective component will be that in the direction OP. The total force in the direction OP due to the charge on the circular strip in question will be—

$$\int_{b=o}^{b=2\pi x}\frac{\sigma\delta x\cos\theta}{\epsilon r^2}\,db=\frac{\sigma\delta x\ \cos\ \theta\ 2\pi x}{\epsilon r^2}\text{ dynes.}$$

The total force at P due to the whole charged surface will be the sum of the forces due to all the elementary strips such as *lm* into which the whole surface of the disc AB may be divided, and the direction of the force will be along OP.

Hence the total force due to the charged surface of AB will be—

$$\int_{\theta=o}^{\theta=\frac{\pi}{2}}\frac{\sigma\ \cos\ \theta\ 2\pi x}{\epsilon r^2}dx=\int_{\theta=o}^{\theta=\frac{\pi}{2}}\frac{2\pi\sigma\ \sin\ \theta d\theta}{\epsilon}\text{ dynes,}$$

since $$\sin\ \theta=\frac{x}{r}$$

and $$\frac{rd\theta}{dx}=\cos\theta.$$

The total force due to the charged surface of AB is therefore—

$$\frac{2\pi\sigma}{\epsilon}\text{ dynes.}$$

The radius of the disc is assumed to be very large compared with the distance a.

Further, it is easy to see that this will be very approximately the value of the force at points in between the plates other than those on the axis, if the distance from the point considered to the edges of the discs is great compared with the distance between the two discs. It will also be apparent that the force will be the same if the plate is not round so long as the distance from the point to the edge of the plate is great compared with the distance from the point to the surface of the plate.

Hence, unit positive charge placed anywhere in between the two charged plates, but not near the edges, will be repelled from the plate AB by a force of $\frac{2\pi\sigma}{\epsilon}$ dynes.

By precisely similar reasoning it is seen that unit positive charge will be attracted by the charged surface of the plate CD by a force of $\frac{2\pi\sigma}{\epsilon}$ dynes, so that the total force on unit charge in between the two plates will be—

$$\frac{4\pi\sigma}{\epsilon}\text{ dynes.}$$

It follows, therefore, that the lines of force between two such charged surfaces will be parallel and perpendicular to the plane of the plates except at places near the edges, as shown in Fig. 28.

27. Electrostatic Pressure, or the Electric Force on each Element of Charge on a Conductor due to the remainder of the Charge.—Consider two points M and M^1 (Fig. 29), such that M is close to the outside surface of a closed charged conductor and M^1 is close to M, but on the inside of the surface.

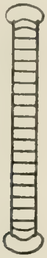

Fig. 28.

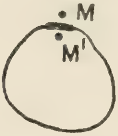

Fig. 29.

The total intensity F of the force at the point M is made up of the force f due to the charge on the element of surface δS near M, and the force f^1 due to the charge on the rest of the surface. Let σ electrostatic c.g.s. units per sq. cm. be the density of the charge on the element of surface δS.

Then the force F at M is by § 23—

$$F = f + f^1 = \frac{4\pi\sigma}{\epsilon} \text{ dynes.}$$

The total force at M^1 just inside the surface is zero. But the force f^1 due to all the charged surface other than δS is the same at M^1 as at M, and the force f at M^1 due to the element δS has only changed its sign.

Hence, for the point M^1 just inside the charged surface, the force F is—

$$f^1 - f = o;$$

that is

$$f = f^1;$$

therefore

$$2f = \frac{4\pi\sigma}{\epsilon} \text{ dynes,}$$

or

$$f^1 = f = \frac{2\pi\sigma}{\epsilon} \text{ dynes.}$$

The force on the charged element δS due to all the charged surface other than the element is—

$$f^1\sigma\delta S \text{ dynes;}$$

that is, the force on the charge $\sigma\delta S$ of the element is—

$$\frac{2\pi\sigma^2\delta S}{\epsilon} \text{ dynes;}$$

or, considering an element of surface of 1 sq. cm. area, the force is—

$$\frac{2\pi\sigma^2}{\epsilon} \text{ dynes per sq. cm.}$$

$$=\frac{\epsilon F^2}{8\pi} \text{ dynes per sq. cm.}$$

This is called the **electrostatic pressure** at the part of the charged conductor considered. Since the value of the electrostatic pressure is proportional to the square of the density it is always in the same direction—*i.e.*, is directed outwards whether the charge is positive or negative.

For the atmosphere under normal conditions of temperature and barometric pressure, the limiting value of the electrostatic pressure is about 640 dynes per sq. cm., this value corresponding to a potential gradient of 38,000 volts per cm.

If the electrostatic pressure at the surface of a charged conductor in air, at normal temperature and barometric pressure, exceeds about 640 dynes per sq. cm., the air ceases to insulate and the charge begins to leak from the conductor (see also p. 72 and § 60).

CHAPTER III

POTENTIAL—CAPACITY

28. Potential.—Consider two charged and insulated conductors. If one end of a wire be allowed to touch one conductor and the other end of the wire to touch the other conductor, electricity will generally pass along the wire from one conductor to the other, and it will be found that the wire becomes heated by the passage of the electricity. The energy corresponding to the heat generated in the wire will have been derived from the electric field.

It may happen, however, that when the two charged conductors are connected by a wire no transfer of electricity takes place, and when this is the case the two conductors are said to be at the same *electric potential.*

If electricity is transferred from one conductor to the other along the connecting wire the two conductors are said to be at different electric potentials, or, in other words, there is a *potential difference* between the conductors before being joined by a wire (see § 46, B).

It is clear, therefore, that a charged conducting surface in equilibrium must be everywhere at the same potential, otherwise electricity would pass along the surface from the place at the higher potential to the place at the lower potential until the potential became equalised over the whole surface.

The potential of a conductor depends not only on the quantity of electricity with which it is charged, but also on the geometrical shape and the presence of other charges in the neighbourhood.

The potential difference between two charged conductors is measured by the amount of energy which would be required to transfer positive electricity from the conductor at the lower potential to the conductor at the higher potential (see p. 58).

There is said to be unit difference of potential between two charged conductors if unit energy is required to transfer unit positive charge from the conductor at the lower potential to the conductor at the higher potential. It is assumed that the transfer of the unit positive charge produces no appreciable effect on the electrical state of the bodies.

In the electrostatic c.g.s. system of units, if one erg of work is expended in transferring one electrostatic c.g.s. unit of positive electricity from conductor B to conductor A, the conductor A is said to have a higher potential than the conductor B, and the potential difference is one electrostatic c.g.s. unit.

For many practical purposes this electrostatic c.g.s. unit of potential difference is too large, and the practical unit adopted is the **volt.**

One electrostatic c.g.s. unit of potential difference = 300 volts (see § 292).

29. The Work done in transferring Electricity from a Point at a High Potential to a Point at a Lower Potential depends solely on the Potential Difference between the Two Points and the Quantity of Electricity transferred, but does not depend on the Path followed during the Transfer.—Assume that the point A (Fig. 30) is at a higher potential than the point B, then if unit positive charge of electricity is taken from A to B along the path ACB, the amount of work done by the electrical field will be equal to the potential difference between A and B. Now let the unit charge be taken back from B to A along the path BDA. Then the charge will have arrived back at the starting-point, and consequently the original conditions of the field have been re-established. It follows, therefore, that energy must have been restored to the electric field by an amount precisely equal to that which was absorbed from the field in transferring the unit charge from A to B along the path ACB, and this is true whatever path be chosen for the return journey from B to A. That is to say, the work done by the electric field in moving a unit positive charge from a point at potential V_1 to a point at potential V_2 is $V_1 - V_2$, and is independent of the path taken by the charge during the transfer.

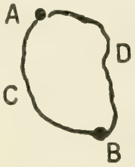

Fig. 30.

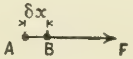

Fig. 31.

30. The Electric Force in any Direction is the Rate of Decrease of Potential in that Direction.—In Fig. 31 let the electric force at the point A be F and have the direction shown. Consider a point B close to A, and let δx be the distance of B from A. In moving one electrostatic c.g.s. unit of positive electricity from A to B the amount of work done by the electric field will be—

$$F\delta x \text{ ergs.}$$

But this is the amount δV by which the potential of A is greater than the potential of B. Hence—

$$-\delta V = F\delta x;$$

the minus sign before δV indicating that the potential diminishes in the direction A to B.

Therefore $$F = -\frac{dV}{dx}.$$

Hence, the electric intensity of force at A is equal to the rate of change of potential at that point, and is in the direction corresponding to maximum rate of diminution of potential.

31. Zero Potential.—For convenience of reference a body is said to have zero potential if it is uncharged and at an indefinitely great distance from any other charged conductor.

It follows that the potential of a charged conductor is the work done in bringing unit positive charge from an indefinitely great distance to that conductor, assuming that the introduction of the unit charge does not appreciably influence the electrification of the conductor in question. If the field due to the charged conductor is relatively a weak field the potential might be measured by bringing, say, $\frac{1}{1000}$ unit of positive electricity from an indefinitely great distance to the conductor, the potential of the conductor in this case being then 1,000 times the work done in the transfer.

If work is done by the electric field in transferring the unit positive charge from an indefinitely great distance to the conductor the potential of the conductor is negative.

For practical purposes the earth is taken to be at zero potential.

32. Potentials of Conductors in Various Cases—*Case I.—An insulated charged sphere at an indefinitely great distance from other bodies.*

Let Q electrostatic c.g.s. units be the charge on the sphere, and let r cms. be the radius of the sphere (Fig. 32).

It has already been shown in § 21 that the electric force at points outside the sphere is the same as if the whole charge of the sphere were concentrated at the centre.

FIG. 32.

The electric force at a point P distant x cms. from the centre of the sphere is—

$$\frac{Q}{\epsilon x^2} \text{ dynes.}$$

The work done in moving one electrostatic c.g.s. unit of positive electricity from the point P a distance δx towards the sphere is therefore—

$$\frac{Q}{\epsilon x^2}\,\delta x \text{ ergs.}$$

Hence the work done in bringing unit positive charge from an infinite distance to the surface of the sphere is—

$$\int_{\infty}^{r} \frac{Q}{\epsilon x^2}\,dx \text{ ergs}$$

$$=\frac{Q}{\epsilon r} \text{ ergs,}$$

and this is the potential of the charged sphere measured in electrostatic c.g.s. units.

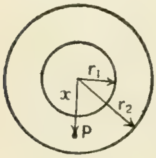

FIG. 33.

This is also the potential at a point distant r cms. from a concentrated charge Q, § 34.

Case II.—A conducting sphere totally enclosed in a concentric spherical conducting shell.

In Fig. 33 let r_1 cms. be the radius of the sphere and let r_2 cms. be the radius of the inner surface of the concentric spherical shell.

Let Q electrostatic c.g.s. units be the charge on the sphere, the charge on the inner surface of the spherical shell being consequently $-Q$ (see § 1).

The force at any point P distant x from the common centre is—

$$\frac{Q}{\epsilon x^2} \text{ dynes,}$$

due to the charge on the sphere.

The force at P due to the charge on the outer spherical shell is, of course, zero (see § 14).

Hence the potential difference between the sphere and the spherical shell is—

$$\int_{r_2}^{r_1} \frac{Q}{\epsilon x^2}\,dx;$$

that is— $$\frac{Q}{\epsilon}\left(\frac{1}{r_1}-\frac{1}{r_2}\right) \text{ electrostatic c.g.s. units.}$$

Case III.—The potential inside a charged conductor.

The electric force inside a charged conductor is zero. From § 30 it follows that—

$$\frac{dV}{dx}=o,$$

hence $$V=\text{a constant,}$$

or the potential at points inside a charged conductor is everywhere constant, and is equal to the potential of the conductor.

Case IV.—The potential difference between two parallel plates charged with equal and opposite quantities of electricity, the distance between the plates being small compared with the surface dimensions of the plates.

In § 26 it was shown that the lines of force between two such plates are, except near the edges of the plates, straight parallel lines perpendicular to the plates. It has also been proved that the electric force in between the plates is $\frac{4\pi\sigma}{\epsilon}$ dynes for points which are not close to the edges, σ being the surface density of the charge.

The potential difference between the plates is therefore—

$$\frac{4\pi\sigma}{\epsilon} d$$

$$=\frac{4\pi Q}{\epsilon S} d \text{ electrostatic c.g.s. units.}$$

where S sq. cms. is the area of the charged face of each plate, d cms. is the distance between the opposing faces of the plates, and Q the total charge on each plate in electrostatic c.g.s. units. The fringing effect near the edges of the plates is assumed to be relatively negligibly small.

The expression $\frac{4\pi\sigma}{\epsilon} d$ for the potential difference between the plates where σ is the surface density of the charge at a comparatively large distance from the edges is, however, exact, notwithstanding the fringing effect.

33. Equipotential Surfaces.—The lines of force at the surface of a charged conductor in equilibrium are perpendicular to the surface of the conductor. If not, there would be a force tangential to the surface of the conductor tending to move the charges over the surface, and the distribution of electricity would accordingly be altered until a steady state had been reached such that the lines of force were perpendicular to the surface. Further, from the definition of potential difference it follows that the surface of such a charged conductor must everywhere be at the same potential, otherwise electricity would be transferred from the part of the surface at a higher potential to a part of the lower potential, until a steady uniform potential had been reached.

A surface drawn in an electric field such that the potential at every point of the surface is the same is called an **equipotential surface.** It is obvious that an equipotential surface must be everywhere at right angles to the lines of force, for since the potential V is constant over the surface, the force is—

$$F = -\frac{dV}{dx} = o,$$

where x is measured in a direction tangential to the surface at the point considered.

34. Potential due to Several Charges.—Suppose in the neighbourhood of a given point P there are several electrified bodies—A, B, C . . . with charges Q_1, Q_2, Q_3 . . . respectively; and suppose that each body is so small compared with its distance from P that the charge on the body may be considered as being concentrated at a point in the body. The resultant potential at P will then be the algebraical sum of the potentials at P due to each individual charged body—*i.e.*,

$$V = V_1 + V_2 + V_3 + \ . \ . \ . = \left(\frac{Q_1}{r_1} + \frac{Q_2}{r_2} + \frac{Q_3}{r_3} + \ . \ . \ .\right),$$

where r_1, r_2, r_3 . . . are the respective distances of the charged bodies from the point P.

Generally, the distribution of electricity in the neighbourhood of the point cannot be taken as being concentrated at one point in the body or at a finite number of separate points. In such cases it is necessary to divide all the charges into an infinite number of separate elemental charges δQ and to find the value of $\Sigma \frac{\delta Q}{r}$ for each charged body.

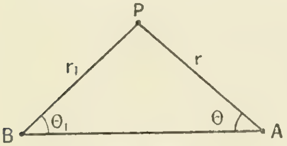

FIG. 34.

Example 1.—Suppose that a charge of $+Q$ units is concentrated at the point A and a charge of $-Q_1$ units at the point B (Fig. 34). The potential at any point P distant r and r_1 cms. respectively from A and B is—

$$\frac{Q}{\epsilon r} - \frac{Q_1}{\epsilon r_1} \text{ electrostatic c.g.s. units.}$$

The surface which satisfies the equation—

$$\frac{Q}{r} - \frac{Q_1}{r_1} = \text{a constant}$$

is an equipotential surface.

If the constant is zero—

$$\frac{Q}{r} - \frac{Q_1}{r_1} = o,$$

and this represents a sphere.

In Fig. 35 the traces in the plane of the paper of a number of equipotential surfaces are shown for the case in which the charge at A is $+5$ units and the charge at B is -1 unit.

The equation for the equipotential surfaces is thus—

$$\frac{5}{r}-\frac{1}{r_1}=\text{a constant.}$$

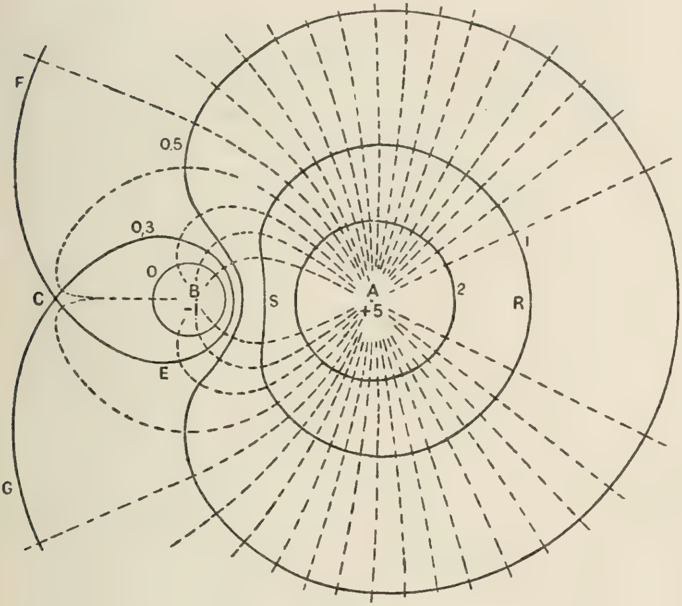

Fig. 35.

The sphere of zero potential is given by the equation—

$$\frac{5}{r}-\frac{1}{r_1}=o;$$

that is, $5\,r_1=r,$

and the trace of this surface is shown by the circle O, of which the centre is at a point to the left of B.

The equipotential surfaces are given by the revolution of the system shown in Fig. 35 about the axis AB. Each equipotential line shown in this figure will then generate an equipotential surface.

All points inside the circle O are at negative potential.

In Fig. 35 the lines of force between the charges on A and B are also drawn, the equation for the lines of force being—

$$5\cos\theta+\cos\theta_1=\text{a constant.}$$

Compare Figs. 15 and 35.

At the point C—viz., such that

$$\frac{5}{CA^2} = \frac{1}{CB^2},$$

the force is zero. The equipotential surface through this point forms a loop. For the particular case chosen—viz., the distance AB is 5 cms., and the charges + 5 and − 1 at A and B respectively, the potential of this surface is 0·3 electrostatic c.g.s. units. For positive potentials lower than this value the equipotential surfaces will be two distinct surfaces, one inside the loop CE and the other completely outside the surface CFG. For points at a higher potential than 0·3 the equipotential surfaces will pass between the loop CE and the point A, as, for example, the surface corresponding to the curve SR.

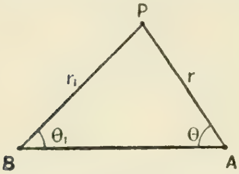

Fig. 36.

Example 2.—In Fig. 36, suppose that a charge of one unit of positive electricity is concentrated at A and one unit of positive electricity at the point B. The equipotential surfaces for this case will be given by the equation—

$$\frac{1}{r} + \frac{1}{r_1} = \text{a constant.}$$

The traces of the surfaces drawn in this way are shown in Fig. 37 for the case of + 2 units at A and + 2 units at B.

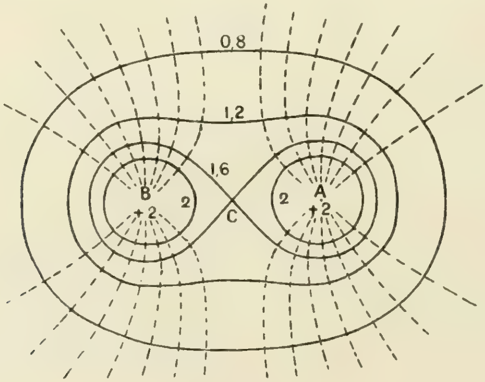

Fig. 37.

The lines of force are given by the equation—

$$\cos\theta - \cos\theta_1 = \text{a constant,}$$

and the lines of force are shown drawn dotted in Fig. 37, and are also shown in Fig. 18. The point C is midway between A and B, and the equipotential surface through this point forms a double symmetrical loop, the potential of points on this loop being 1·6 electrostatic c.g.s. units, for the case in which the distance AB is 5 cms. For points at a higher potential than 1·6 electrostatic c.g.s. units, the equipotential surfaces are two separate surfaces—viz., one in each part of the double loop through C.

If a thin conducting sheet be imagined to replace any equipotential surface, nothing will be altered as regards the distribution of the lines of force. Generally, however, when a conducting surface is introduced into an electric field it will cut across different equipotential surfaces, and since the conducting surface so introduced must eventually become an equipotential surface, the distribution of the electric charges of the field will be altered to conform to the requirement that the surface introduced may become an equipotential surface.

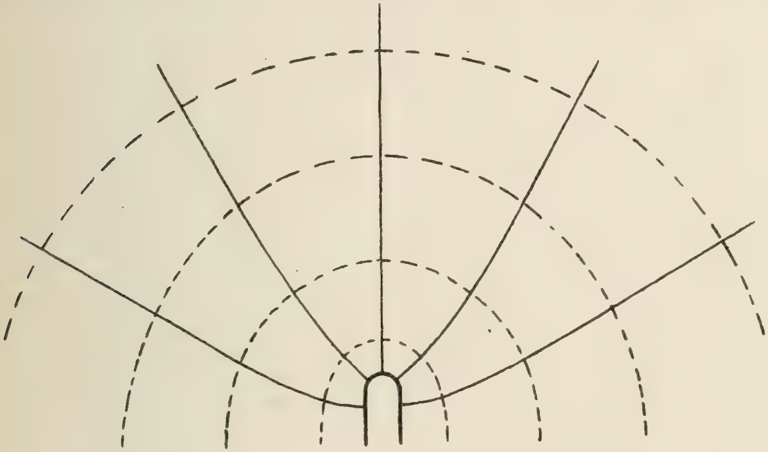

Fig. 38.

35. The Surface Density of a Charged Isolated Conductor is greatest near Projecting Edges or Points.—This may be seen from the following general considerations:

Imagine a charged metal rod isolated in space (Fig. 38). At a great distance from the rod the equipotential surfaces will be spherical, and the lines of force will have such a direction that they will appear to come from the centre of the spherical equipotential surfaces. In other words, at a great distance from the charged conductor the charge acts as though it were concentrated at a point.

Again, the surface of the charged conductor itself is of course an equipotential surface. Hence, as the charged rod is approached from a great distance away the equipotential surfaces gradually change in shape from spherical surfaces to the shape of the rod. The lines of force, however, must everywhere be perpendicular to the equipotential surfaces, and must change in direction as the rod is approached in order to conform with the changing shape of the equipotential surfaces. By drawing the approximate equipotential surfaces in this way it is easy to draw the general direction of the lines of force, and it will be found that they will tend to concentrate at the ends of the rod.

An example of the practical application of the effect of a pointed conductor in concentrating the charge at a point is found in lightning conductors. When a charged cloud comes in the neighbourhood of the point of the lightning conductor a charge is induced on the lightning conductor by the charged cloud, and the surface density at the point of the conductor becomes very great, and eventually the insulating layer of air between the cloud and the conductor is ruptured, a spark passes, and the cloud becomes discharged.

36. Capacity of an Electric Field.—In practice the electric fields most frequently met with are those in which the boundaries, which form the ends of the lines of force, are conducting surfaces. The potential difference between the boundaries of such a field is $\int F dx$ where F is the intensity at any point in the field, and the integration is performed along a line of force from one boundary to the other. The value of F, the intensity at any point in the field, is, by Coulomb's Law, directly proportional to the quantity of electricity on each boundary and therefore the value of the potential difference between the boundaries is directly proportional to the charge on each of the two boundaries. Hence, we may write—

$$\int F dx = V = \frac{Q}{C},$$

or

$$C = \frac{Q}{V},$$

where C is a constant for any given configuration of the boundaries and any given insulating medium between the boundaries. This constant is called the **capacity** of the field formed by the two boundary surfaces and the insulating medium.

Hence *the capacity of an electric field is defined as the ratio of the quantity of the charge on each boundary surface to the potential difference between the surfaces.*

When two electric conductors are so placed that their surfaces form the boundaries of a field of relatively great capacity, the arrangement is usually called a **condenser.** An example of a condenser is the arrangement shown in Fig. 33, § 32. A Leyden jar is also a form of condenser.

A condenser has a capacity of one electrostatic c.g.s. unit if one electrostatic c.g.s. unit of quantity of electricity on each boundary (*i.e.*, a positive unit on one boundary and a negative unit on the other boundary) raises the potential difference of the boundaries by one electrostatic c.g.s. unit.

For practical purposes the unit employed is the **farad** and its subdivision the **micro-farad,** which is one-millionth part of a farad.

An electric field has a capacity of one farad if one coulomb of electricity on each boundary (*i.e.*, a coulomb of positive electricity on one boundary and a coulomb of negative electricity on the other boundary) produces a potential difference between the boundaries of one volt.

Since 1 coulomb $=3\times10^9$ electrostatic c.g.s. units, and 300 volts $=1$ electrostatic unit, it follows that

1 farad $=9\times10^{11}$ electrostatic c.g.s. units,
1 micro-farad $=9\times10^5$ electrostatic c.g.s. units.

37. Capacity of a Tube of Force.—Consider a tube of force stretching from one boundary of conducting surface δS_1 to the other boundary of conducting surface δS_2 (Fig. 39). The quantity of electricity at each end of the tube of force is—

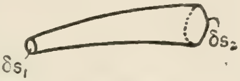

Fig. 39.

$$q=\sigma_1\delta S_1=\sigma_2\delta S_2,$$

where σ_1 is the surface density at δS_1 and σ_2 the surface density at δS_2.

If F_1 is the intensity of the force at δS_1 and F_2 the intensity of the force at δS_2, it follows from § 23 that—

$$F_1=\frac{4\pi\sigma_1}{\epsilon}:\ F_2=\frac{4\pi\sigma_2}{\epsilon}:$$

therefore $$q=\frac{F_1\epsilon}{4\pi}\delta S_1=\frac{F_2\epsilon}{4\pi}\delta S_2.$$

The potential difference between the ends of the tube of force—*i.e.*, between the surfaces δS_1 and δS_2—is

$$\int F\,dx,$$

the integration being taken from one boundary surface to the other.

The quantity of electricity on each boundary is—

$$q=\frac{\epsilon F_1\delta S_1}{4\pi}=\frac{\epsilon F_2\delta S_2}{4\pi}=\frac{\epsilon F\delta S}{4\pi},$$

where F is the electric intensity at any point in the tube, and δS is the cross-sectional area of the tube at that point. Since the areas δS, δS_1, δS_2 are at right angles to the lines of force of the tube, each of these areas must be a part of equipotential surfaces, cutting the tube at the respective positions.

Hence the capacity of the tube is—

$$C = \frac{q}{\int F\,dx} = \frac{F_1 \epsilon \delta S_1}{4\pi \int F\,dx} = \frac{F_1 \epsilon \delta S_1}{4\pi F_1 \delta S_1 \int \frac{dx}{\delta S}};$$

that is

$$C = \frac{\epsilon}{4\pi} \frac{1}{\int \frac{dx}{\delta S}}.$$

The capacity of an element ABCD (Fig. 40) of the tube is therefore—

$$\frac{\epsilon}{4\pi} \frac{1}{\frac{\delta x}{\delta S}},$$

where δx is the length of the element measured along the lines of force and δS is the mean cross-sectional area of the element—*i.e.*, δS is the element of the equipotential surface EF intercepted by the tube.

If δx is taken sufficiently small—*i.e.*, the boundary surfaces AB and CD sufficiently close together—δS may be taken as the element of the equipotential surface forming either end of the element of the tube of force—viz., the area of either AB or CD.

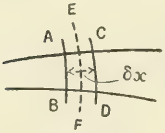

Fig. 40.

In the following paragraphs the capacity of the electric field for some important practical cases will be determined.

38. Capacity of an Isolated Spherical Conductor.—In § 32 it was shown that the potential of a sphere of radius r cms. charged with Q units of electricity is—

$$V = \frac{Q}{\epsilon r}.$$

Hence the capacity is $C = \frac{Q}{V} = \epsilon r$ electrostatic c.g.s. units.

That is, the capacity in electrostatic c.g.s. units of a spherical conductor at a great distance from other conductors is equal to the radius of the sphere in cms. multiplied by the dielectric constant ϵ.

If the sphere is in air the value of ϵ is unity and the capacity is—

$$C = r \text{ electrostatic c.g.s. units;}$$

that is, $C=\frac{r}{9\times 10^5}$ micro-farads (see § 36).

39. Capacity of a Sphere totally enclosed in a Concentric Spherical Conducting Shell.—Let r_1 cms. be the radius of the sphere and r_2 cms. the radius of the inner surface of the concentric shell (Fig. 41). In § 32 it was shown that if a quantity of electricity Q is given to the sphere the potential difference between the surfaces of sphere and shell will be—

$$\frac{Q}{\epsilon}\left(\frac{1}{r_1}-\frac{1}{r_2}\right).$$

Hence the capacity is—

$$C=\frac{Q}{V}$$

$$=\frac{\epsilon r_1 r_2}{r_2-r_1} \text{ electrostatic c.g.s. units,}$$

or $$C=\frac{\epsilon r_1 r_2}{9\times 10^5(r_2-r_1)} \text{ micro-farads.}$$

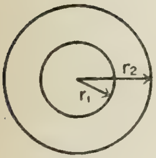

FIG. 41.

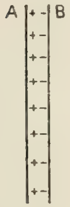

FIG. 42.

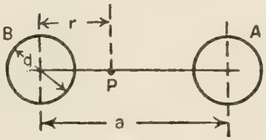

FIG. 43.

40. The Capacity of the Field formed between Parallel Conducting Plates at a Small Distance apart.—In § 32 the potential difference between the two plates when a charge + Q is given to one plate and a quantity − Q is on the other plate is—

$$\frac{4\pi Q}{\epsilon S}d,$$

where d cms. is the distance between the charged faces of the plates (where d is a small quantity), and S sq. cms. is the area of the charged face of each plate (Fig. 42).

Hence the capacity of the field between the two plates is—

$$\frac{\epsilon S}{4\pi d} \text{ electrostatic c.g.s. units;}$$

that is, $$C=\frac{\epsilon S}{113\times 10^5 d} \text{ micro-farads.}$$

41. Capacity of the Field between two Long Thin Cylinders arranged with their Axes Parallel—*Case I.—Suppose the dis-*

tance apart a *cms.* (*Fig.* 43) *is great compared with the diameter* d *cms. of each cylinder.*

The charge on each cylinder may then be assumed to be concentrated on the axis of the cylinder. This is equivalent to the assumption that the density of the charge on each cylinder is uniform over the surface.

Let $+q$ be the charge in electrostatic c.g.s. units per unit length of cylinder A, and let $-q$ be the density of charge per unit length of cylinder B.

The force at a point P in between the cylinders is, by § 25,

$$\frac{1}{\epsilon}\left[\frac{2q}{r}+\frac{2q}{a-r}\right] \text{ dynes.}$$

Hence the work done in moving unit charge from the surface of one cylinder to the surface of the other is—

$$\int_{r=\frac{d}{2}}^{r=a-\frac{d}{2}} \frac{1}{\epsilon}\left[\frac{2q}{r}\,dr+\frac{2q\,dr}{a-r}\right]$$

$$=\frac{4q}{\epsilon}\log_e\left[\frac{2a-d}{d}\right] \text{ ergs;}$$

that is, the potential difference between the cylinders is, § 30—

$$V=\frac{4q}{\epsilon}\log_e\left[\frac{2a-d}{d}\right] \text{ electrostatic c.g.s. units.}$$

The quantity of the charge per unit length of each cylinder is q, and hence the capacity is—

$$C=\frac{q}{V}$$

$$=\frac{\epsilon}{4\log_e\left[\frac{2a-d}{d}\right]} \text{ electrostatic c.g.s. units per cm length.}$$

If d is small compared with a, the capacity is very approximately—

$$C=\frac{\epsilon}{4\log_e\frac{2a}{d}} \text{ electrostatic c.g.s. units per cm, length.}$$

It is usually more convenient to work with logarithms to base 10. The logarithm to base e is equal to 2·3 times the logarithm to base 10. If this conversion is made and if the capacity is expressed in practical units (*i.e.*, micro-farads) instead of electrostatic units the above results may be written—

$$C = \frac{\epsilon}{51{\cdot}4 \log_{10} \frac{2a}{d}} \text{ micro-farads per mile of double cable,}$$

where d is small compared with a. For conductors in air $\epsilon = 1$, see also Section V., §§ 256 and 257.

The lines of force between the charged cylinders may be determined as follows:

In Fig. 44 suppose the cylinders are represented by the traces of their axes—viz., A and B.

Let the distance of the point C from A be r cms., and the distance of the point C from B be r_1 cms. Let the angle CAB be θ and the angle CBA be θ_1.

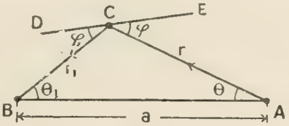

FIG. 44.

Suppose the charge per unit length of the cylinder A be + 1, and the charge per unit length of the cylinder B be − 1. The force at C due to the charge on A will be $\frac{2}{\epsilon r}$ dynes in the direction AC, and the force at C due to the charge on B will be $\frac{2}{\epsilon r_1}$ dynes in the direction CB as was shown in § 25.

Let DCE be the direction of the line of force through C. The component of the force perpendicular to DCE must be zero. Let the angle BCD be ϕ_1 and the angle ACE be ϕ.

Then
$$\frac{2}{\epsilon r_1} \sin \phi_1 = \frac{2}{\epsilon r} \sin \phi;$$

but
$$\frac{r_1 d\theta_1}{ds} = \sin \phi_1 : -\frac{r d\theta}{ds} = \sin \phi,$$

where ds is a small element of length along the line of force, reckoned positive in the direction from A towards B.

Therefore
$$\frac{d\theta_1}{ds} + \frac{d\theta}{ds} = o,$$

or
$$\theta_1 + \theta = \text{a constant}.$$

Hence the lines of force are circles passing through A and B, the centres of the circles being on the line which bisects AB at right angles. These lines of force are marked F, F in Fig. 45. The lines of force between the cylinders will therefore lie on cylindrical surfaces, of which the traces in the plane of the paper are the circles F, F in Fig. 45.

For the equipotential lines in the plane of the paper the condition to be fulfilled is that the force along the lines is zero (see § 33). If now ECD represents an equipotential line, then—

$$\frac{2}{\epsilon r_1} \cos \phi_1 - \frac{2}{\epsilon r} \cos \phi = o.$$

But $$\cos \phi_1 = -\frac{dr_1}{ds} : \cos \phi = \frac{dr}{ds}.$$

Therefore $$-\frac{1}{r_1} \frac{dr_1}{ds} + \frac{1}{r} \frac{dr}{ds} = o;$$

or $$\log_e \left[\frac{r}{r_1}\right] = \text{a constant};$$

that is, $$r = \text{a constant} \times r_1.$$

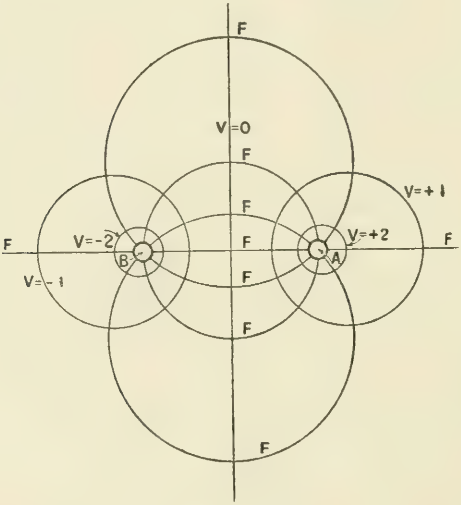

Fig. 45.

These are circles V, V, whose centres lie on the line AB, and cut the circles F, F, at right angles (Fig. 45). The conditions for Fig. 45 are: AB$=5$ cms., $\epsilon=1$; the charge q per cm. length on A is $+\frac{1}{2}$ unit and on B is $-\frac{1}{2}$ unit.

Case II.—The capacity of the field of two long parallel cylindrical conductors of which the diameter is comparable with the distance between the axes of the cylinders.

Let d cms. be the diameter of each cylinder, and let a cms. be the distance between their axes.

It is first noted that the surfaces of the cylinders are equipotential surfaces, and the equation for these circles may be written—

$$r_1 = A\, r,$$

where A is a constant, and $M : M_1$ are the points from which the distances r and r_1 are measured, and $O : O_1$ are the centres of the circles. The distance between the centres of the cylinders is a cms., and the distance between the points $M : M_1$ is b cms. It is easy to show that

$$b=\sqrt{a^2-d^2},$$

d being the diameter of each cylinder (Fig. 46).

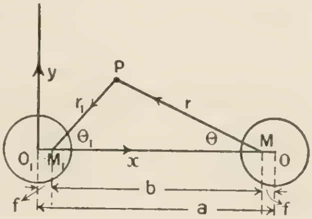

FIG. 46.

This may be proved as follows:

Suppose O_1 is taken as the centre of Cartesian co-ordinates. Then for points on the circle whose centre is at O_1—

$$x^2+y^2-2x\left[\frac{f-A^2(f+b)}{1-A^2}\right]=\frac{A^2(f+b)^2-f^2}{1-A^2}.$$

But since the origin is at O_1, the coefficient of x must be zero—*i.e.*,

$$f=A^2(f+b),$$

or

$$A^2=\frac{f}{f+b},$$

and the radius of the circle is—

$$\frac{d}{2}=\sqrt{\frac{A^2(f+b)^2-f^2}{1-A^2}}=\sqrt{f^2+fb}.$$

Therefore

$$f^2+fb-\frac{d^2}{4}=o,$$

and

$$2f=(a-b).$$

Hence

$$\frac{(a-b)^2}{4}+\frac{(a-b)b}{2}-\frac{d^2}{4}=o;$$

or

$$b=\sqrt{a^2-d^2}.$$

Now for equipotential lines of which the equation is—

$$r_1=A\,r,$$

the corresponding lines of force will be given by the equation—

$$\theta_1+\theta=\text{a constant},$$

as may be seen from the case previously dealt with. These are circles passing through M and M_1. It follows, therefore, that the effect of two oppositely charged cylinders of diameter d, and whose axes are distant a apart, is the same as if the charges were concentrated on lines whose traces are shown at M and M_1 in Fig. 46, the distance between M and M_1 being—

$$\sqrt{a^2-d^2}.$$

From this it is seen that if d is small compared with a the points M and M_1 very nearly coincide with the points O and O_1 respectively, as was assumed to be the case previously.

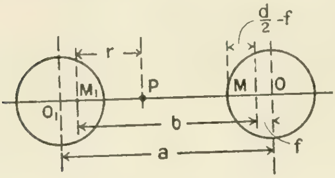

FIG. 47.

The capacities of the cylinders for the case when $\frac{d}{a}$ is not small may now be deduced by the aid of these results.

The force at P (Fig. 47) is

$$\frac{2q}{\epsilon\, r}+\frac{2q}{\epsilon\,(b-r)} \text{ dynes.}$$

The work done in moving unit charge from the surface of one conductor to the surface of the other is—

$$\int_{r=\frac{d}{2}-f}^{r=b-\left(\frac{d}{2}-f\right)} \frac{1}{\epsilon}\left[\frac{2q\,dr}{r}+\frac{2q\,dr}{b-r}\right]$$

$$=\frac{2q}{\epsilon}\left[\log_e\left(b-\frac{d}{2}+f\right)^2-\log_e\left(\frac{d}{2}-f\right)^2\right]$$

$$=\frac{4q}{\epsilon}\log_e\frac{b-\frac{d}{2}+f}{\frac{d}{2}-f}$$

$$=\frac{4q}{\epsilon}\log_e\left[\frac{\sqrt{a^2-d^2}-(d-a)}{\sqrt{a^2-d^2}+(d-a)}\right],$$

since $b^2=a^2-d^2$, and $2f=a-b$.

Hence the capacity per cm. length of cable is—

$$C=\frac{q}{V}=\frac{\epsilon}{4\log_e\left[\frac{\sqrt{a^2-d^2}-(d-a)}{\sqrt{a^2-d^2}+(d-a)}\right]},$$

or $$C=\frac{\epsilon}{4\log_e\left[\frac{\sqrt{a^2-d^2}+a}{d}\right]}$$ electrostatic c.g.s. units per cm. length.

If the logarithm base be changed from e to 10, and the capacity expressed in practical units, then

$$C=\frac{\epsilon}{51\cdot 4\log_{10}\left[\frac{\sqrt{a^2-d^2}+a}{d}\right]}$$ micro-farads per mile of double cable.

If $\frac{d}{a}$ is small this formula reduces to that given for Case I, p. 45.

For the case of cables in air $\epsilon=1$. See also Section V., §§ 256 and 257.

42. The Capacity of the Field of a Concentric Cable.—In Fig. 48 let d cms. be the diameter of the inner cable and D cms. the diameter of the inner surface of the outer cable. The lines of force between the two cables when the inner is charged with a quantity of electricity Q per unit length will be radial. The force at a point distant x cms. from the common centre is $\frac{2Q}{\epsilon x}$ (see § 25). The potential difference between the inner and the outer cab is therefore—

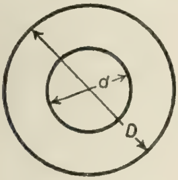

FIG. 48.

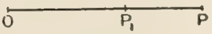

FIG. 49.

$$\int_{\frac{d}{2}}^{\frac{D}{2}} \frac{2Q}{\epsilon x}dx=\frac{2Q}{\epsilon}\log_e\frac{D}{d} \text{ electrostatic c.g.s. units.}$$

Hence the capacity per cm. length will be—

$$C=\frac{\epsilon}{2\log_e\frac{D}{d}} \text{ electrostatic c.g.s. units,}$$

or $$C=\frac{\epsilon}{25{\cdot}7\log_{10}\frac{D}{d}} \text{ micro-farads per mile of concentric cable.}$$

43. The Capacity of the Field between a Long Horizontal Wire and the Earth.—As a preliminary step in dealing with this case, some important geometrical propositions will be given.

If from a fixed point O (Fig. 49) a radius vector OP is drawn to any curve, and in OP a point P_1 is taken such that

$$OP\times OP_1=\text{a constant,}$$

the curve traced out by the point P_1 is said to be the *inverse* of that on which P lies. The point O is called the *centre of inversion*, and the constant the *constant of inversion*.

Proposition 1.—*The inverse of a circle with reference to a point outside the circle is another circle.*

In Fig. 50 let the given circle be CAB and the centre of inversion be O.

Let C_1 be the inverse point of C and B_1 the inverse point of B with respect to O. Draw any line OQP to cut the circle at Q and P, and let Q_1 and P_1 be the inverse points of Q and P respectively.

Join PB, QC, Q_1C_1.

Then
$$OQ_1 \times OQ = OC_1 \times OC,$$

and
$$\frac{OQ_1}{OC_1} = \frac{OC}{OQ}.$$

Hence the triangles OC_1Q_1 and OQC are similar, and the angle OC_1Q_1 equal to the angle OQC. But the angles CQP plus CBP equal two right angles. Therefore the angle OQC equals the angle OBP, and the angle OC_1Q_1 equals the angle OBP.

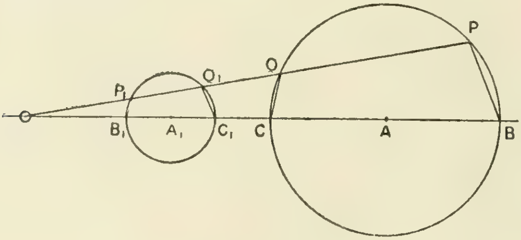

FIG. 50.

The triangles OC_1Q_1 and OBP are therefore similar, and the point Q_1 traces out a curve similar to that traced out by the point P, which is a circle.

Similarly it may be proved that the inverse of a sphere with reference to a point outside the sphere is another sphere.

If it is required to find the position of O so that the circle inverts into itself—*i.e.*, so that the circle BPC inverts into the circle $B_1P_1C_1$ (Fig. 51)—draw the tangent OP. The inverse of the point P will be P_1, a coincident point.

Then
$$OB \times OB_1 = OP^2 = \text{a constant, say } k^2.$$

Hence
$$OA^2 = AP^2 + OP^2 = R^2 + k^2.$$

Thus the position of the centre of inversion O is found for any given value of the constant of inversion k^2.

Proposition 2.—The inversion of a circle with reference to a point on the circumference of the circle is a straight line. The inverse of a sphere, with reference to a point on its circumference, is a plane.

In Fig. 52 let OAB be the circle and let O be the centre of inversion. Let B_1 be the point inverse to B. Take any other point C on the circle and let C_1 be the inverse point.

Dec 30

Then $$OC \times OC_1 = OB \times OB_1.$$

Also the angle OCB is a right angle. Hence the angle OB_1C_1 is a right angle. Therefore C_1 moves on a straight line at right angles to OB.

It may be proved in a similar way that the inverse of a sphere with reference to a point on the circumference is a plane.

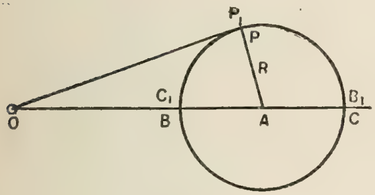

Fig. 51.

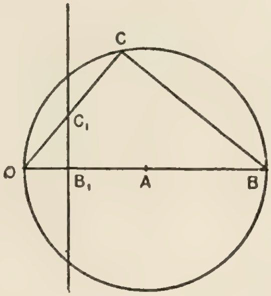

Fig. 52.

Proposition 3.—*If there is a pair of curves such as the circles ED and CB, Fig.* 53, *and if these curves be inverted with reference to a point O, the value of* $\frac{SW}{SK}$ *for an elementary area SWYK of the first pair of curves will be the same as the value of* $\frac{S_1W_1}{S_1K_1}$ *for the area* $S_1W_1Y_1K_1$ *of inversion of the second pair of curves.*[*]

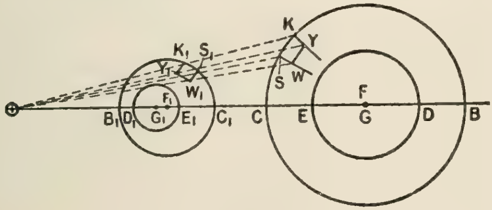

Fig. 53.

This may be proved as follows:

$$OS \times OS_1 = OW \times OW_1 = k^2.$$

Therefore the triangles OS_1W_1 and OWS are similar,

and $$\frac{SW}{S_1W_1} = \frac{OW}{OS_1}.$$

Similarly $$\frac{SK}{S_1K_1} = \frac{OK}{OS_1}.$$

---

[*] See Steinmetz, *Elektrotechnische Zeitschrift*, 1893.

Therefore
$$\frac{SK}{SW}=\frac{OS_1\times S_1K_1\times OK}{OW\times S_1W_1\times OS_1}$$
$$=\frac{S_1K_1\times OK}{S_1W_1\times OW}=\frac{S_1K_1}{S_1W_1}$$

if $OK=OW$, as will be the case when the dimensions of the area SWYK are indefinitely small.

Now suppose that BC and ED are the traces of two cylindrical conductors, and suppose SK and WY are equipotential lines, and SW and KY are lines of force. The sum of the values of $\frac{SK}{SW}$ for all such elements as SWYK will be a measure of the capacity of the field between the two cylindrical conductors (see § 41). Hence it follows that the capacity of the field between the conductors ED and CB is the same as the capacity of the field between the conductors E_1D_1 and C_1B_1, which are inverted from the original circles with reference to any centre of inversion O. *This result enables the capacity of the field between a horizontal wire and the earth to be found in a very simple manner.*

In Fig. 54 let the wire be represented by the circle BAC and the earth by the line DE. Let d cms. be the diameter of the wire and h cms. the height of the axis of the wire from the earth.

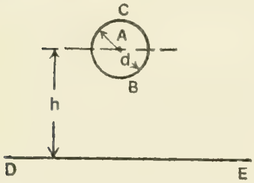

Fig. 54.

Now invert the circle BAC and the line DE so that the circle BAC inverts into itself, and the line DE inverts into a circle concentric with BAC. From what has been said above the capacity of the field between these two concentric circles will be the same as the capacity of the field between the cylinder BAC and the plane DE in Fig. 54. Since the capacity of the two concentric cables is easily found as in § 42 the problem is at once solved.

In Fig. 55 let D be the diameter of the circle OD_1E_1, which is the inverse of the line DE. The centre of inversion O must be on the circle OD_1E_1 by Proposition 2, above.

Also $\left(\frac{D}{2}\right)^2=OA^2=\frac{d^2}{4}+k^2$ (see Proposition 1 above),

and $OF\times OF_1=k^2$.

That is $\left(\frac{D}{2}-h\right)D=\left(\frac{D}{2}\right)^2-\left(\frac{d}{2}\right)^2.$

From which it follows that $\frac{D}{2}=h+\sqrt{h^2-\frac{d^2}{4}}$

and the required capacity is therefore—

$$C=\frac{\epsilon}{2\log_e \frac{D}{d}};$$

or $$C=\frac{\epsilon}{2\log_e \dfrac{2h+2\sqrt{h^2-\frac{d^2}{4}}}{d}} \text{ electrostatic c.g.s. units per cm. length of cable.}$$

If d is small compared with h this reduces to—

$$C=\frac{\epsilon}{2\log_e \frac{4h}{d}} \text{ electrostatic c.g.s. units per cm. length of cable;}$$

or $$C=\frac{\epsilon}{25{\cdot}7\log_{10} \frac{4h}{d}} \text{ micro-farads per mile of cable.}$$

If these results be compared with those of the case considered in § 41 (Fig. 43) it will be seen that the capacity per unit length of cable of the field between a horizontal cable of diameter d and the earth is twice that of the field between two parallel cables each of diameter d, and of which the distance apart is equal to twice the height of the horizontal cable above the earth (see Fig. 56).

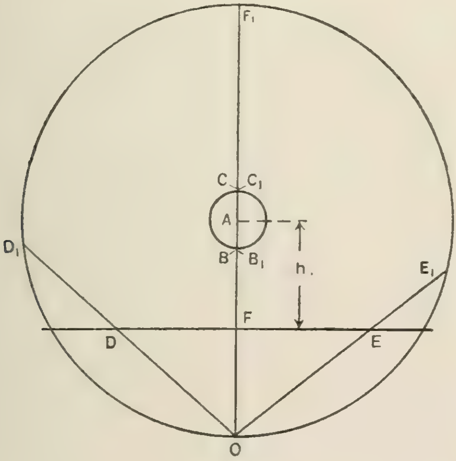

FIG. 55.

The cable *bac* is said to be the **electrical image** of the cable BAC in the surface DE.

44. The Capacity of the Field between Two Eccentric Cables.—In Fig. 57 let FG and OC represent the two eccentric cables.

From what has been said (§ 43) it is clear that the capacity of the field between these two cables is the same as the capacity of the field of the system formed by inverting the two cables with reference to any centre of inversion.

Choose the centre of inversion on the circumference of the circle OC, and choose the constant of inversion so that the circle FG inverts into itself. The circle OC will invert into a straight line LKM perpendicular to the line OF. Hence the capacity of the field between the two eccentric cables OC and FG will be the same as the capacity of the field between the cylinder FG and the plane LKM. It was proved in § 43 that the capacity of the field between a cable and a conducting plane parallel to the axis of the cable and at a distance h from the axis is equal to twice the capacity of the field between the cable and a precisely similar cable arranged parallel to it, and such that the distance between the axes is $2h$.

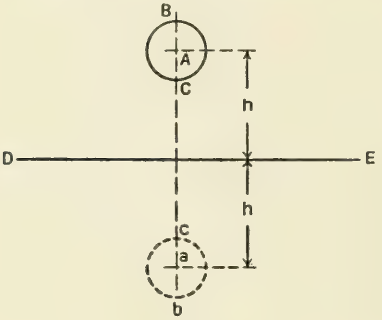

FIG. 56.

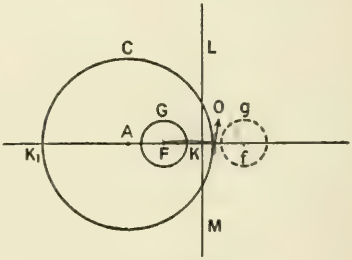

FIG. 57.

Hence for the purpose of finding the capacity of the field between the two eccentric cables OC and FG in Fig. 57, the equivalent system of the two cables FG and fg in Fig. 57 may be used, and the capacity may be written down at once from the results deduced in § 43.

If in Fig. 57, x is the distance OF, then h the distance between the axis F and the plane LM into which the circle OC inverts may be deduced as follows:

$$\mathrm{OF}^2=\frac{d^2}{4}+k^2,$$

where d is the diameter of the cable FG.

Also $$\mathrm{OK}\times\mathrm{OK}_1=k^2.$$

Hence $$\mathrm{OK}=\frac{\mathrm{OF}^2-\frac{d^2}{4}}{\mathrm{OK}_1}=\frac{x^2-\frac{d^2}{4}}{\mathrm{D}},$$

where D is the internal diameter of the cable OC.

Therefore $$\mathrm{FK}=\mathrm{OF}-\mathrm{OK};$$

that is, $$\mathrm{FK}=x-\left[\frac{x^2-\frac{d^2}{4}}{\mathrm{D}}\right],$$

and the capacity of the field between the eccentric cables is—

$$\mathrm{C}=\frac{\epsilon}{2\log_e\left[\frac{2\mathrm{FK}+\sqrt{4\mathrm{FK}^2-d^2}}{d}\right]}$$ electrostatic c.g.s. units per cm. length of cable;

or $$\mathrm{C}=\frac{\epsilon}{25{\cdot}7\log_{10}\left[\frac{2\mathrm{FK}+\sqrt{4\mathrm{FK}^2-d^2}}{d}\right]}$$ micro-farads per mile.

For further reference to the results given in §§ 41, 42, 43, and 44, see Section V., §§ 256 and 257.

45. The Capacity of a Condenser formed of Two Parallel Plates separated by Air, and having a Slab of Insulating Material of Specific Inductive Capacity ϵ inserted between the Plates.—It is again assumed here that the distance between the plates is small compared with the superficial dimension of each plate.

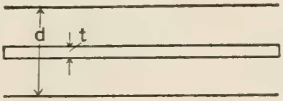

Fig. 58.

Let $+\sigma$ be the surface density of the charge on one plate, and $-\sigma$ the surface density of the charge on the other plate (Fig. 58).

The intensity of the field in the air will be $4\pi\sigma$ (see § 26).

The intensity of the field in the slab of insulating material of specific inductive capacity ϵ inserted between the plates will be $\frac{4\pi\sigma}{\epsilon}$, § 26.

The potential difference between the plates is—

$$\mathrm{V}=4\pi\sigma(d-t)+\frac{4\pi\sigma}{\epsilon}t$$
$$=4\pi\sigma\left[d-\frac{\epsilon-1}{\epsilon}t\right],$$

where d cms. is the distance between the plates and t cms. the thickness of the inserted slab.

The capacity of the field is—

$$C=\frac{Q}{V}=\frac{\sigma S}{4\pi\sigma\left[d-\frac{\epsilon-1}{\epsilon}t\right]}=\frac{S}{4\pi\left[d-\frac{\epsilon-1}{\epsilon}t\right]}\text{ electrostatic c.g.s. units,}$$

where S sq. cms. is the area of the charged surface of the plate.

The capacity of the condenser when the slab is removed is—

$$\frac{S}{4\pi d}\text{ electrostatic c.g.s. units.}$$

Hence, introducing a slab of thickness t is equivalent to reducing the distance between the plates by an amount $\frac{\epsilon-1}{\epsilon}t$.

This result is the basis of one method of determining the value of ϵ for solid substances.

46. Systems of Condensers—A. *Condensers arranged in Series*—If there is a number of condensers A:B:C . . . so arranged that one coating of A is connected to one coating of B, the other coating of B connected to one coating of C, and so on, the condensers are said to be connected in series. Diagrammatically, such an arrangement is shown in Fig. 59.

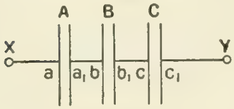

Fig. 59.

Suppose that when the condensers are connected up each is uncharged, and let the series be charged, for example, by connecting the terminals XY to the terminals of an electrical machine (see Chapter V.).

Let the coating a of condenser A be charged with a quantity of electricity $+Q$. The other coating a_1 will be charged with a quantity $-Q$, and since the coating b of condenser B is in conducting connection with a_1 the charge on b_1 will be $+Q$ (see § 3). Similarly the charge on b_1 will be $-Q$, and so on. In other words, each condenser is charged with the same quantity of electricity Q.

Let $C_A:C_B:C_C$ . . . be the capacities of the condensers, $V_A:V_B:V_C$ . . . the potential differences of the coatings of the condensers, and let V be the potential difference of the series arrangement of condensers. Then—

$$V_A=\frac{Q}{C_A}:V_B=\frac{Q}{C_B}:V_C=\frac{Q}{C_C}:$$

$$V=V_A+V_B+V_C+\ldots:$$

Hence $$V=Q\left(\frac{1}{C_A}+\frac{1}{C_B}+\frac{1}{C_C}+\ldots\right),$$

or $$V=\frac{Q}{C},$$

where C is the capacity of the condenser which is equivalent to the series arrangement.

Hence the capacity C of the condenser, which is equivalent to the condensers of capacities $C_A : C_B : C_C \ldots$ respectively and connected in series, is given by the expression—

$$\frac{1}{C} = \frac{1}{C_A} + \frac{1}{C_B} + \frac{1}{C_C} + \ldots$$

B. *Condensers arranged in Parallel*—The condensers A : B : C : . . . are connected in parallel if one coating of each condenser is connected to a common connection F and the other coating of each condenser is connected to a common connection G. Such an arrangement is shown diagrammatically in Fig. 60.

Let V be the potential difference between the points F and G. Let $Q_A : Q_B : Q_C : \ldots$ be the respective charges of the condensers and $C_A : C_B : C_C : \ldots$ the respective capacities.

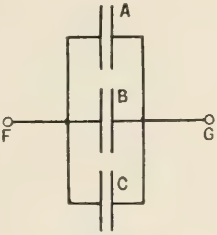

FIG. 60.

Then $$V = \frac{Q_A}{C_A} = \frac{Q_B}{C_B} = \frac{Q_C}{C_C}.$$

The total charge Q of the parallel arrangement of condensers is the sum of the charges of the separate condensers—viz.,

$$Q = Q_A + Q_B + Q_C + \ldots,$$

or $$Q = V(C_A + C_B + C_C + \ldots),$$
$$= VC;$$

where C is the capacity of the condenser which is equivalent to the parallel arrangement. Hence the capacity C of the condenser which is equivalent to the parallel arrangement of condensers of capacities $C_A : C_B : C_C : \ldots$ is given by the expression

$$C = C_A + C_B + C_C + \ldots$$

Now suppose two condensers A and B of capacities C_A and C_B are charged with quantities of electricity Q_A and Q_B respectively. The energy of condenser A is, § 50—

$$\frac{1}{2}\frac{Q_A^2}{C_A},$$

and the energy of condenser B is—

$$\frac{1}{2}\frac{Q_B^2}{C_B}.$$

Suppose the condensers are now connected in parallel. The total charge Q will be $Q_A + Q_B$ and, as has already been proved, the capacity of the combination will be $C_A + C_B$.

The energy of the charge of the parallel connection is therefore—

$$\frac{1}{2}\frac{(Q_A+Q_B)^2}{C_A+C_B}.$$

The loss of energy when connecting in parallel is therefore

$$\frac{1}{2}\left[\frac{Q_A{}^2}{C_A}+\frac{Q_B{}^2}{C_B}\right]-\frac{1}{2}\frac{(Q_A+Q_B)^2}{C_A+C_B}$$

$$=\frac{1}{2}\left[\frac{Q_A{}^2}{C_A}+\frac{Q_B{}^2}{C_B}-\frac{Q_A{}^2+2Q_AQ_B+Q_B{}^2}{C_A+C_B}\right]$$

$$=\frac{1}{2}\frac{(Q_AC_B-C_AQ_B)^2}{C_AC_B(C_A+C_B)}.$$

This expression becomes zero when

$$Q_AC_B=C_AQ_B;$$

that is, when

$$\frac{Q_A}{C_A}=\frac{Q_B}{C_B},$$

i.e., when the potential differences of condenser A and condenser B before connection in parallel are the same. In all other cases the energy of the condensers after connection in parallel is less than the total energy of the two condensers before being connected in parallel. The loss of energy is accounted for by the spark which passes and the heating of the connections when the condensers are connected in parallel (see § 28).

CHAPTER IV

THE DIELECTRIC AND THE ENERGY OF THE ELECTRIC FIELD

47. Specific Inductive Capacity.—Faraday found that the capacity of a condenser depends not only on the size, shape, and distance apart of the conducting surfaces, but also on the nature of the insulating medium between the surfaces.

Two spherical condensers were made as nearly as possible alike, and in one an opening was arranged so that melted wax or sulphur could be run into the space in between the boundary surfaces (Fig. 61).

In the first experiment the space *A* in both condensers was filled with air. A charge was given to one condenser, and the fixed ball of the torsion balance allowed to touch the knob *a*. The two outer coatings were then connected together and earthed. The two inner coatings were also connected together, and the fixed ball of the torsion balance again allowed to touch the knob *a* of the condenser. From the readings of the torsion balance in each case it was found that the charge of the condenser X after the charge had been shared with the condenser Y was half the original charge.

The insulating space A_Y of condenser Y was then filled with the substance, such as sulphur, whose insulating properties were to be examined, and the same process of charging and sharing the charge repeated. It was then found that the share of the charge taken by the condenser Y (sulphur filled) was much greater than that taken by condenser X (air filled), thus showing that the capacity of the sulphur-filled condenser was greater than that of the air-filled condenser.

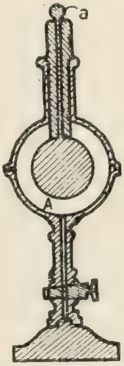

FIG. 61.

Let the two outer surfaces of the condensers X and Y be connected together, the common connection of these surfaces being earthed, and therefore at zero potential. Let the space A_X of condenser X be air filled, and the space A_Y of condenser Y filled with some other insulating material, say sulphur.

Suppose that the charge Q be given to the inner conductor of condenser X, the potential being thereby raised to the value V_1. This potential V_1 may be measured by the torsion balance when the fixed ball of the balance is allowed to touch the knob a_X of the condenser X.

Hence $$V_1 = \frac{Q}{C_X},$$

where C_X is the capacity of the air-filled condenser X.

Now let the knobs $a_X : a_Y$ of the two condensers be connected by means of a conductor. The condenser X will thus share its charge with the condenser Y, and the potential of the inner surfaces of the condensers will thereby attain a common value, say V_2. The reading of the torsion balance when the fixed ball of the balance is brought into contact with the knobs a_X or a_Y, after they have been connected together, is a measure of the value of the potential V_2.

Let q_X now be the charge of the condenser X and q_Y the charge of the condenser Y. Then the following relations will hold:

$$V_1 = \frac{Q}{C_X} : V_2 = \frac{q_X}{C_X} = \frac{q_Y}{C_Y},$$

$$Q = q_X + q_Y : q_X = V_2 C_X : q_Y = V_2 C_Y.$$

Therefore $$Q = V_2 (C_X + C_Y),$$

or $$V_2 = \frac{Q}{C_X + C_Y}.$$

Hence $$\frac{V_1}{V_2} = 1 + \frac{C_Y}{C_X},$$

or $$\frac{C_Y}{C_X} = \frac{V_1}{V_2} - 1 = \epsilon,$$

V_1 and V_2 being given by the readings of the torsion balance, as already explained.

In this way Faraday showed that the introduction of a solid dielectric increased the capacity of a condenser as compared with the capacity of the condenser when air was the dielectric.

The name **specific inductive capacity** was given by Faraday to that property of an insulating medium on which the capacity of a condenser depends.

The ratio:

$$\frac{\text{Capacity of condenser with any given dielectric}}{\text{Capacity of same condenser with air as dielectric}}$$

is the specific inductive capacity of the given dielectric.

The value of the constant ϵ which has already appeared so frequently in previous pages is the specific inductive capacity of the material in question. (See, for example, §§ 7, 26, 48, 55, etc.)

If the dielectric to be examined was a gas, the space A was first exhausted by screwing the foot to an air-pump, and the gas afterwards introduced.

When examining solid dielectrics the two hemispheres could be parted, and for convenience Faraday filled only the lower hemisphere with the solid. Consequently,

$$\frac{C_Y}{C_X}=\frac{1+\epsilon}{1+1}=\frac{1+\epsilon}{2},$$

C_Y being as before, the capacity of the condenser having the solid dielectric.

48. The Dielectric Constant ϵ.—The value of ϵ is constant for any given insulating medium, but has different values for different media. In order to form some kind of mental picture of the mechanism which could account for the action of one electric charge on another at some distance away, Faraday looked for the explanation in the action of the insulating medium itself. It was stated in § 3 that equal quantities of positive and negative electricity may be considered as existing in conductors which are uncharged. It is now to be assumed that in every part of the insulating medium between two charged conductors equal quantities of positive and negative electricity exist, which normally—*i.e.*, in the absence of impressed electric forces—neutralise the effect of each other. When any part of the insulating medium becomes a part of an electric field of force, the positive and negative electricities are separated to some extent, and remain separated so long as the electric force is maintained. When, however, the electric force disappears, the positive and negative electricities in that part of the insulating medium resume their normal relative positions and neutralise the effect of each other.

Thus, consider a tube of force bounded by the two conducting surfaces A and B on which equal charges of positive and negative electricity reside (Fig. 62). The lines of force pass from the positive charge to the negative charge across the intervening insulating medium. If it is now assumed that the medium contains equal quantities of positive and negative electricity, the action of the electric force is to separate out the positive and negative electricity to a very slight degree, the tendency being for the two kinds of electricity to return to their normal relative positions—*i.e.*, **the dielectric is strained.**

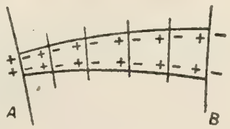

FIG. 62.

If the medium is divided up into cellular spaces by planes perpendicular to the axis of the tube of force—*i.e.*, equipotential surfaces—it is supposed that each end of each cellular space is charged with opposite kinds of electricity, the end of the cell contiguous to the positively charged conducting surface A having a negative charge, and the end of the cell contiguous to the negatively charged conducting surface B having a positive charge, as indicated in Fig. 62.

The amount of the charge developed is thus dependent on the nature of the medium, and is a measure of the value of the specific inductive capacity ϵ. If the medium were a conductor ϵ would be indefinitely large.

The charge developed in the medium at the surface of each charged conductor tends to neutralise the charge on the surface of the conductors; the force at any point in the medium will be smaller the higher the value of ϵ for that medium (see §§ 9, 26, 55, etc.).

Substance.	ϵ.
Air (or vacuum)	1
Silk	1·6
Manila paper	1·8
Petroleum	2·1
Paraffin wax	2·32
Pure rubber	2·34
Impregnated paper	3 to 4
Vulcanised rubber	3 to 5 (about)
Vulcanised bitumen	see p. 216
Ebonite	3·15
Sulphur	3·84
Porcelain	4·38
Mica	5
Light flint glass	6·6
Dense flint glass	9·9
Pure water	78·5

49. Residual Charge.—Suppose a condenser with a dielectric of glass—say a Leyden jar—be charged and allowed to stand a short while, and then discharged by means of discharging tongs—*i.e.*, by allowing a spark to pass between the coatings. The two coatings are thus brought to the same potential, and the jar appears to be completely discharged. If, however, the jar is allowed to stand a short while with one coating insulated it is found that a charge gradually accumulates of the same sign as the original charge, and a second spark may be obtained on discharging by means of the discharging tongs, this second spark being much smaller than the first. If the jar be again insulated and allowed to stand a short time a third spark may be obtained, and the process may even be repeated a third or fourth time. The successive charges after the first discharge are known as *residual charges.*

Further, if a condenser which has a marked residual effect be charged to a known potential difference and then left insulated for some time, the potential difference will be found gradually to fall, and if the condenser be then discharged the quantity of electricity which passes in the discharge is found to be less than that of the original charge.

These residual effects of certain dielectrics complicate the problem of determining the specific inductive capacities of such materials. The difficulties may be largely obviated by making the measurement of the specific inductive capacity immediately after charging.

Extremely good insulators such as air show no such residual effects, and consequently condensers with air as the dielectric form good standard condensers.

50. Energy of the Electric Field.—Suppose that the boundary surfaces A and B of an electric field (Fig. 63) have potential difference V. This field may have been assumed to have been established by transferring the quantity $+Q$ of electricity from the boundary B to the boundary A.

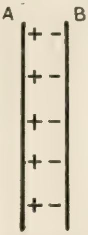

FIG. 63.

When the potential difference is V, suppose that a further quantity of electricity δQ is transferred from B to A, the quantity being so small that the potential difference between the boundaries is not appreciably affected by the transfer. By definition of potential difference the work done in making this transfer is—

$$V\delta Q \text{ ergs.}$$

If C is the capacity of the field and Q is the charge when the potential difference is V, all the quantities being measured in electrostatic c.g.s. units, then

$$Q=VC,$$

or

$$V=\frac{Q}{C}.$$

Hence the work done in effecting the transfer of the total quantity of electricity Q from the boundary B to the boundary A—*i.e.*, in establishing the field—is

$$\int_0^Q V dQ=\int_0^Q \frac{Q}{C}\, dQ$$

$$=\tfrac{1}{2}\,\frac{Q^2}{C}$$

$$=\tfrac{1}{2}\,VQ \text{ ergs.}$$

There is good reason to believe that this energy is stored in the dielectric (see §§ 49 and 51).

The above results may also be arrived at as follows: Suppose the values of Q and V be plotted as co-ordinates, then since

$$\frac{Q}{V}=C \text{ and is constant,}$$

the points plotted for corresponding values of Q and V will be a straight line passing through the origin O.

At the time when the quantity is given by aQ_1 (Fig. 64), suppose that this is increased by an amount δQ to bQ_2 (Fig. 64). The work done in making this change is—

$$V\delta Q = \text{area } aQ_1Q_2b.$$

Hence the total energy stored when the quantity is Q=BC (Fig. 64) is the area OBC—that is

$$\tfrac{1}{2}\ QV \text{ ergs},$$

where Q is the charge and V is the potential difference due to this charge, both in electrostatic c.g.s. units.

This leads to a simple representation of the energy of an electric field.

Suppose that the field is mapped out by tubes of force, such that each tube starts from a unit of quantity on the positive boundary, and ends on a unit of quantity on the negative boundary. In this way the field will be completely divided into tubes of force, such that the number of the tubes will give the numerical value of the charge.

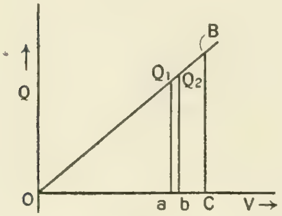

Fig. 64.

Suppose now equipotential surfaces be drawn, starting from one boundary as one surface, and the surfaces so drawn that the potential difference between any two consecutive surfaces is unity. Each tube of force will then be divided into a set of compartments or cells, the number of which will give the value of the potential difference between the boundary surfaces. Hence, the number of tubes of force will give the value of the total quantity Q, and the number of cells in each tube will give the value of the potential difference V between the boundary surfaces.

Hence, the total number of cells into which the field is divided in this way is QV, or twice the numerical value of the total energy of the field.

If one or both boundaries of the field were capable of collapsing, the work done by the field during the approach of the boundaries would be equal to the energy stored in the field. Since the potential difference V would be proportional to the distance between the boundaries (see § 32), the potential difference would gradually fall to zero as the boundaries met. The mean potential difference would thus be—

$$\tfrac{1}{2}\ V,$$

and the work done—

$$\tfrac{1}{2}\ QV \text{ ergs}.$$

Further, since the number of cells into which the field has been divided is QV, the energy stored in each of the cells is $\frac{1}{2}$ erg (see § 54).

51. The Electric Energy of a Condenser is stored in the Dielectric.—Suppose a Leyden jar, provided with removable inner and outer coatings, is charged and placed on an insulating support. Now let the glass jar be removed from the two coatings, the whole of the parts being kept insulated. On bringing the two metal coatings together, at most only a slight spark is observed. If, however, after the two metal coatings have been thus discharged, the jar is again assembled, it will give a spark on discharge almost as powerful as if the coatings had not been removed and discharged as described. This experiment proves that the energy of the condenser is stored in the dielectric (see also § 49).

52. The Energy of a Field due to a System of Charges.—Suppose there is a system of conductors A, B, C . . . charged respectively with quantities Q_A, Q_B, Q_C . . ., the corresponding potentials being V_A, V_B, V_C . . .

In the first place it is to be observed that if the charge on each conductor were diminished to $\frac{1}{n}$ of its actual value, the potential of each conductor would also be diminished to $\frac{1}{n}$ of its actual value. This follows by reason of the fact that the potential at any point due to an electric charge is proportional to that charge (see § 36).

Now suppose the system is charged by bringing up the charges to the respective conductors A, B, C . . . in very small quantities δq_A, δq_B, δq_C. . . .

If at any intermediate stage of this process of charging the potentials of the conductors are respectively V_A^1, V_B^1, V_C^1 . . . then the work done in bringing up the extra charges δq_A, δq_B, δq_C . . . at this stage to the respective conductors is—

$$V_A^1\delta q_A + V_B^1\delta q_B + V_C^1\delta q_C + \ . \ . \ . \ \text{ergs},$$

if small quantities of the second order are neglected—that is to say, neglecting the work due to *the increase* of potentials of these conductors produced by these small charges δq_A, δq_B, δq_C . . . as compared with the work done in bringing up these elementary charges against the potential which the conductors had at the beginning of this stage of the charging process.

Hence, the total work in charging the system is—

$$\int_{q_A=0}^{q_A=Q_A} V_A^1\,\delta q_A + \int_{q_B=0}^{q_B=Q_B} V_B^1\,\delta q_B + \int_{q_C=0}^{q_C=Q_C} V_C^1\,\delta q_C + \ . \ . \ . \ \text{ergs}.$$

But $q_A = C_A V_A{}^1$, $q_B = C_B V_B{}^1$, $q_C = C_C V_C{}^1$ . . . where C_A, C_B, C_C . . . are the respective capacities in the field of the conductors A, B, C. . . .

Thus the total work done in charging the system is—

$$\frac{1}{C_A}\int_{q_A=0}^{q_A=Q_A} q_A dq_A + \frac{1}{C_B}\int_{q_B=0}^{q_B=Q_B} q_B dq + \frac{1}{C_C}\int_{q_C=0}^{q_C=Q_C} q_C dq_C + \ldots \text{ ergs}$$

$$= \tfrac{1}{2}\frac{Q_A{}^2}{C_A} + \tfrac{1}{2}\frac{Q_B{}^2}{C_B} + \tfrac{1}{2}\frac{Q_C{}^2}{C_C} + \ldots \text{ ergs}$$

$$= \tfrac{1}{2} V_A Q_A + \tfrac{1}{2} V_B Q_B + \tfrac{1}{2} V_C Q_C + \ldots \text{ ergs,}$$

and this is the energy of the electric field due to the system of charges.

53. The Change in Energy of a System of Charged Conductors due to a Relative Displacement of the Conductors.—Two cases are to be distinguished—viz., (1) The charge on each conductor is constant throughout the displacement, and (2) the potential of each conductor is maintained constant throughout the displacement.

Case I.—The charge on each conductor remains constant throughout the displacement. That is to say, each conductor of the system is completely insulated throughout the displacement.

The conductors will be acted upon by electric forces, and if free to move, each will move in the direction of the resultant force acting on it.

Let the conductors be A, B, C, . . . and the respective charges Q_A, Q_B, Q_C. . . . Let the potentials be V_A, V_B, V_C . . . before the displacement, and $V_A{}^1$, $V_B{}^1$, $V_C{}^1$ . . . after the displacement.

The energy of the system before the displacement is, therefore,

$$\tfrac{1}{2} Q_A V_A + \tfrac{1}{2} Q_B V_B + \tfrac{1}{2} Q_C V_C + \ldots$$

and the energy of the system after the displacement is—

$$\tfrac{1}{2} Q_A V_A{}^1 + \tfrac{1}{2} Q_B V_B{}^1 + \tfrac{1}{2} Q_C V_C{}^1 + \ldots$$

The work done by the electric forces during the displacement is, therefore, by the principle of the Conservation of Energy,

$$\tfrac{1}{2} Q_A(V_A - V_A{}^1) + \tfrac{1}{2} Q_B(V_B - V_B{}^1) + \tfrac{1}{2} Q_C(V_C - V_C{}^1). \ldots$$

Since the system is insulated, the energy corresponding to this work must have been drawn from the energy of the electric field, and hence **the conductors move so that the energy of the electric field diminishes.**

Example 1. Consider the case of two discs parallel to each other, and charged respectively with $+Q$ and $-Q$ electrostatic c.g.s. units and distant d cms. apart (Fig. 67). Let V be the potential difference between them, measured in electrostatic c.g.s. units.

The energy of the field is—

$$\tfrac{1}{2}\,QV \text{ ergs.}$$

The force between the plates tends to draw them together—*i.e.*, to increase the capacity and consequently to *decrease* the potential difference.

Suppose the plates approach under the action of the electric force, and let d^1 cms. be the distance apart after the diplacement, and V^1 the potential difference. The energy of the field is now—

$$\tfrac{1}{2}\,QV^1 \text{ ergs.}$$

The loss of energy is $\tfrac{1}{2}\,Q(V - V^1 \text{ ergs})$, and this is the amount of work expended in drawing the plates towards each other.

Example 2. Consider now two equal spheres, each charged with $+Q$. The potential of each sphere will, therefore, be the same—*i.e.*, the potential difference between them will be zero.

The two spheres will tend to move away from each other, and if they so move, the potential of each sphere will diminish, and consequently the energy of the charge of each sphere will decrease, or the energy of the field will be diminished by the displacement.

Case II.—The potential of each conductor is maintained constant throughout the displacement. For instance, each conductor may be connected to an electric source (battery) which is at constant potential.

Let A, B, C . . . be the conductors maintained respectively at the potentials V_A, V_B, V_C, . . . and suppose the conductors move under the action of the electric forces.

Let Q_A, Q_B, Q_C . . . be the charges on the conductors before the displacement, and $Q_A{}^1$, $Q_B{}^1$, $Q_C{}^1$ . . . the charges after the displacement.

The energy of the charges before the displacement is—

$$\tfrac{1}{2}\,Q_AV_A + \tfrac{1}{2}\,Q_BV_B + \tfrac{1}{2}\,Q_CV_C + \ldots$$

and after the displacement—

$$\tfrac{1}{2}\,Q_A{}^1V_A + \tfrac{1}{2}\,Q_B{}^1V_B + \tfrac{1}{2}\,Q_C{}^1V_C + \ldots$$

The energy gain is, therefore,

$$\tfrac{1}{2}\,(Q_A{}^1 - Q_A)V_A + \tfrac{1}{2}\,(Q_B{}^1 - Q_B)V_B + \tfrac{1}{2}\,(Q_C{}^1 - Q_C)V_C + \ldots$$

The total energy supplied by the sources at constant potential to which the conductors are connected is—

$$(Q_A{}^1 - Q_A)V_A + (Q_B{}^1 - Q_B)V_B + (Q_C{}^1 - Q_C)V_C + \ldots$$

If the conductors move under the action of the electric forces the electric energy supplied to the system will be sufficient to correspond to the increase of the electric energy of the field, plus the mechanical work performed by the electric forces in producing the displacement.

It follows that **the mechanical work performed is equal to the increase of the electric energy of the field, and each is one-half of the total energy supplied from the sources to which the conductors are connected.**

Example.—Suppose two equal spheres A and B are connected respectively to the positive and negative poles of a battery (Fig. 65).

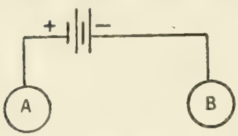

FIG. 65.

Let $+Q_A$ be the charge on A and $-Q_B$ the charge on B, and let $+V_A$ be the potential of A and $-V_B$ the potential of B, measured in electrostatic c.g.s. units. The two spheres will tend to approach each other. Suppose they so move under the action of the electric forces. The total energy of the charges on A and B before the displacement will be—

$$\tfrac{1}{2}\,Q_AV_A + \tfrac{1}{2}\,Q_BV_B \text{ ergs.}$$

The energy of the charges on A and B after the displacement will be respectively—

$$\tfrac{1}{2}\,Q_A{}^1V_A + \tfrac{1}{2}\,Q_B{}^1V_B \text{ ergs.}$$

The increase of the total energy of the system at the end of the displacement will be—

$$\tfrac{1}{2}\,(Q_A{}^1 - Q_A)V_A + \tfrac{1}{2}\,(Q_B{}^1 - Q_B)V_B \text{ ergs.}$$

But the energy supplied by the battery will be—

$$(Q_A{}^1 - Q_A)V_A + (Q_B{}^1 - Q_B)V_B \text{ ergs.}$$

Hence the electrical energy supplied to the system equals twice the increase of electrical energy of the field,

or—

The mechanical work done during the displacement equals the increase of the electric energy of the field.

A further example of this principle is found in the case of the quadrant electrometer dealt with in § 67.

54. Energy stored per Unit Volume of Dielectric.—Let α sq. cms. be the cross-section of a unit tube of force at any place, and let d cms. be the distance between two consecutive equipotential surfaces at the same place in the field. The volume of this cell will thus be—

$$\alpha d \text{ cubic cms.}$$

But the intensity of the force is by § 37—

$$F = \frac{4\pi\sigma}{\epsilon} = \frac{4\pi}{\epsilon\alpha} \text{ dynes,}$$

and $$Fd = \text{potential difference} = 1;$$

therefore $$\alpha d = \frac{4\pi}{\epsilon F^2}.$$

Since αd c.cms. is the volume of the unit cell into which the field may be divided, as explained in § 50, the energy stored in this cell is ½ erg. Hence the energy stored in *unit volume* will be—

$$\frac{1}{2\alpha d} \text{ ergs;}$$

that is— $$\frac{\epsilon F^2}{8\pi} \text{ ergs per c.cm.}$$

This result may be arrived at somewhat differently, as follows:

Let A and B be two discs, uniformly charged respectively with positive and negative electricity of surface density σ, and let the radius of each disc be large compared with the distance between them (Fig. 66).

It follows from what was said in § 26 that the lines of force between the discs will be straight lines parallel to each other, except at places near the edges of the discs. Since the discs are assumed to be large compared with the distance between them, the amount of the fringing of the lines will be negligible compared with the area of the field.

Now, it has been shown that the pull on each unit area of each charged plate is $\frac{2\pi\sigma^2}{\epsilon}$ dynes (see § 27).

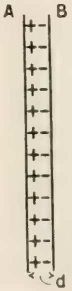

Fig. 66.

Hence the pull on each plate is $\frac{2\pi\sigma^2 S}{\epsilon}$, where S sq. cms. is the area of each plate. That is to say, the force necessary to keep the plates separated is $\frac{2\pi\sigma^2 S}{\epsilon}$ dynes. Suppose now that the plates are moved apart from the distance d cms. to the distance d_1 cms. The work done in this movement is—

$$\frac{2\pi\sigma^2 S}{\epsilon}(d_1 - d) \text{ ergs.}$$

This work is stored up as energy in the electric field, and can be recovered by allowing the plates to move together.

If the plates move together from the distance d cms. to zero the work done will be

$$\frac{2\pi\sigma^2 Sd}{\epsilon} \text{ ergs,}$$

and this is the whole of the energy stored in the field, because when the plates come into contact the charges neutralise one another, and no further work can be obtained. Hence it is seen that the energy stored in the electric system comprising the two charged plates each of area S sq. cms. charged with an electric surface density σ and separated by a distance d cms. is—

$$\frac{2\pi\sigma^2 Sd}{\epsilon}$$

$$=\frac{\epsilon F^2 S}{8\pi} d \text{ ergs,}$$

where F is the intensity of the electric field.

This energy is stored in the volume Sd c. cms. of the dielectric.

This result can be applied to any elementary portion of the dielectric—*i.e.*, the energy stored in the dielectric is—

$$\frac{\epsilon F^2}{8\pi} \text{ ergs per c.cm.}$$

For some numerical examples of the energy stored in the dielectric, see § 57.

55. Electric Strain or Electric Displacement.—It is assumed that the application of the electric force F strains the dielectric in an analogous manner to the strain of a stretched spring (see § 48). Electric energy is thus stored in the dielectric when under the influence of an electric force, just as energy is stored in an extended spring.

For a stretched rod of elastic material the energy stored is—

$$\tfrac{1}{2} \text{ Pull} \times \text{Extension,}$$

or energy stored per unit length is—

$$\tfrac{1}{2} \text{ Pull} \times \frac{\text{Extension}}{\text{Length}}$$

$$=\tfrac{1}{2} \text{ Pull} \times \text{Strain.}$$

The energy stored per unit of volume

$$=\tfrac{1}{2} \text{ Intensity of Pull} \times \text{Strain.}$$

Writing down the expression for the energy stored in the dielectric:

Stored electric energy

$$=\tfrac{1}{2}\ \mathrm{F}\ \frac{\epsilon \mathrm{F}}{4\pi}\ \text{ergs per c.cm. (§ 54).}$$

$$=\tfrac{1}{2}\ \text{Electric Force} \times \text{Electric Strain.}$$

The quantity $\frac{\epsilon \mathrm{F}}{4\pi}$ is termed the **electric strain.** It is also called the **electric displacement** or the **polarisation in the dielectric** in the direction of F, and will be denoted by the letter D.

Further, for the case of a stretched rod, the relationship holds—

$$\frac{\text{Intensity of Pull}}{\text{Strain}}=\text{Modulus of Elasticity.}$$

In the case of the electric field,

$$\frac{\text{Electric Force}}{\text{Electric Strain}}=\frac{\mathrm{F}}{\mathrm{D}}=\frac{\mathrm{F}4\pi}{\epsilon \mathrm{F}}=\frac{4\pi}{\epsilon}.$$

Hence the quantity $\frac{4\pi}{\epsilon}$ is analogous to the modulus of elasticity for elastic materials.

At the surface of a conductor the value of $\frac{\epsilon \mathrm{F}}{4\pi}$ is the surface density σ (§ 37). Moreover, since a tube of strain begins on a positive charge and ends on an equal negative charge the product: strain × cross-sectional area of the tube, is constant throughout the tube (see also §§ 12, 27, and 62).

One unit tube of strain starts from each positive unit of electric quantity.

56. Dielectric Strength.—When an insulator is subjected to an electric force the material becomes electrically strained (§§ 48, 55), and if the electric force exceeds a certain limit, the dielectric becomes mechanically ruptured at some place and an electric discharge takes place, resulting in the dissipation in heat of some or all of the energy stored in the dielectric.

In gases or liquids the rupture is temporary, the insulator becoming restored as soon as the electric force is reduced. In the case of solids, however, the rupture punctures the material and destroys its insulating properties. The electric force which caused rupture of the dielectric is called the dielectric strength of the insulator, and is usually measured in volts or kilo-volts per centimetre of thickness of the material (see also § 27).

Test results on various dielectrics apparently show that the dielectric strength decreases with increase of thickness of the dielectric, and C. Baur* has proposed the following formula connecting the potential difference necessary to rupture a given thickness of insulating material, viz.:

---

* *Electrician*, 1901, vol. xlvii.

$$V = Cx^{2/3},$$

where V is the potential difference necessary for rupture,
x the thickness of the insulating material,
and C is a constant for the particular insulating material.

This formula may also be written,

$$\frac{V}{x} = \frac{C}{\sqrt[3]{x}},$$

or, according to this formula, the dielectric strength in volts per cm. of thickness is inversely proportional to the cube root of the thickness.

The real significance of this rule, however, seems to be that the tests of dielectric strength are dependent on the variation of intensity of the electric field in which the insulating material is placed. If, for example, the test electrodes between which the sample is placed are needle points, the intensity of the electric force in the neighbourhood of the points is much greater than at some intermediate distance (§§ 27 and 35), and the tests measure the *mean value* of the stress over the given thickness of insulation. The thicker the sample, however, the greater will be the difference between the maximum and the mean values, and hence the apparent diminution of the dielectric strength as the thickness increases. For very thin samples the mean value more nearly approaches the maximum value, and hence the true dielectric strength will be more nearly given by tests on thin samples. In practice it is usual to allow a factor of safety of about 4 or 5 to take into account possible local irregularities of surface which tend to concentrate the electric intensity (see also § 58, Fig. 69).

Tests made by Gray, using as electrodes two polished copper discs which were portions of spheres of 70 cms. diameter, the edges of the discs being rounded, gave the following results:

Material.	*Thickness of Sample in Cms.*	*Dielectric Strength in Volts per Cm. of Thickness.*
Air	1·0	29,800
Crystal glass	0·1	285,000
	0·5	183,000
Window glass	0·2	160,000
Mica	0·01	1,150,000
	0·1	610,000
Micanite	0·05 0·1	400,000

Material.	*Thickness of Sample in Cms.*	*Dielectric Strength in Volts per Cm. of Thickness.*
Sheet rubber	0·135 0·27	476,000 318,000
Ebonite	0·093 0·186	538,000 434,000
Raw linseed oil	0·4 to 0·8	83,000
Boiled linseed oil	do.	85,000
Vaseline oil	do.	60,000
Light mineral lubricating oil	do.	48,000
Paraffin oil	—	64,000 to 101,000

Results for other materials obtained by various investigators* are as follows:

Material.	*Dielectric Strength in Volts per Cm. of Thickness.*
Gutta Percha	109,000
Paraffin wax (solid)	130,000 to 270,000
,, ,, (melted)	56,000
Resin oil	270,000 to 1,350,000
Vulcanized rubber	100,000
Empire cloth	125,000
Manila paper	430,000
Vulcanized bitumen	see p. 216

In the case of air it has been shown that there is a minimum sparking potential difference—viz., about 790 volts—below which it is impossible to obtain a discharge.

The true dielectric strength of air—that is, the dielectric strength in a uniform field—is about 38,000 volts per cm. for normal conditions of temperature and atmospheric pressure.

In the case of gases the pressure of the gas has a marked effect on the dielectric strength. In fact, for air, the dielectric strength increases almost directly proportionally to the pressure.†

57. Maximum Value of the Energy which can be Stored in a Dielectric.—In § 54 it was shown that the energy stored in a dielectric is—

$$\frac{\epsilon F^2}{8\pi} \text{ ergs per c.cm.}$$

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* See C. Baur, *Electrician*, 1901, vol. xlvii.; M. O'Gorman, *Journal of Inst. of Elec. Eng.*, vol. xxx.

† See E. A. Watson, *Journal of Inst. of Elec. Eng.*, vol. xliii., 1909.

But the electric intensity F is equal to the rate of diminution of potential in the direction of F (see § 30) or

$$F = -\frac{dV}{dx} \text{ dynes,}$$

where V is in electrostatic units and the length x is in cms.

If V is expressed in volts, then since 1 electrostatic unit of potential difference is equal to 300 volts (see § 28),

$$F = -\frac{1}{300}\frac{dV}{dx} \text{ dynes,}$$

or

$$F = \frac{\text{fall of potential in volts per cm.}}{300} \text{ dynes.}$$

Hence the energy stored per c.cm. of the dielectric is—

$$\frac{\epsilon F^2}{8\pi} = \frac{\epsilon}{9 \times 10^4 \times 8\pi} (\text{fall of potential in volts per cm.})^2 \text{ ergs}$$

$$= \frac{\epsilon}{9 \times 10^{11} \times 8\pi} (\text{fall of potential in volts per cm.})^2 \text{ joules}$$

$$= \frac{0{\cdot}737\epsilon}{9 \times 10^{11} \times 8\pi} (\text{fall of potential in volts per cm.})^2 \text{ ft. lbs.,}$$

since 1 joule $= 10^7$ ergs $= 0{\cdot}737$ foot pounds (see Section VI., p. 472).

Hence the maximum value of the energy which can be stored in a dielectric is—

$$\frac{0{\cdot}737\epsilon}{9 \times 10^{11} \times 8\pi} (\text{dielectric strength in volts per cm.})^2 \text{ ft. lbs. per c.cm.,}$$

or

$$0{\cdot}53 \times 10^{-12}\epsilon\, (\text{dielectric strength in volts per cm.})^2 \text{ ft. lbs. per cubic inch.}$$

Examples.—For air at normal temperature and pressure the dielectric strength is 38,000 volts per cm. and the specific inductive capacity ϵ is unity. Hence the maximum amount of energy which can be stored in air at normal temperature and pressure is—

$$0{\cdot}53 \times 10^{-12} \times (38000)^2 \text{ ft. lbs. per cubic inch}$$
$$= 7{\cdot}66 \times 10^{-4} \text{ ft. lbs. per cubic inch.}$$

For mica the dielectric strength is about 1,150,000 volts per cm., and the specific inductive capacity ϵ is about 5. Hence the maximum amount of energy which can be stored in mica is about

$$0{\cdot}53 \times 10^{-12} \times 5 \times (1{\cdot}15 \times 10^6)^2 \text{ ft. lbs. per cubic inch}$$
$$= 3{\cdot}5 \text{ ft. lbs. per cubic inch.}$$

58. Electric Stresses in Cable Dielectrics.—In Fig. 67 let the radius of the core of the cable be r cms. and the inside radius of the outer sheath be R cms. Then if ϵ is the specific inductive capacity of the dielectric, the capacity of the cable per 1 cm. of length will be—

$$C=\frac{\epsilon}{2\log_e\frac{R}{r}}$$

electrostatic units (see § 43).

If the core of the cable is charged to a potential V electrostatic c.g.s. units by a charge of q electrostatic c.g.s. units per 1 cm. length, then—

$$q=VC.$$

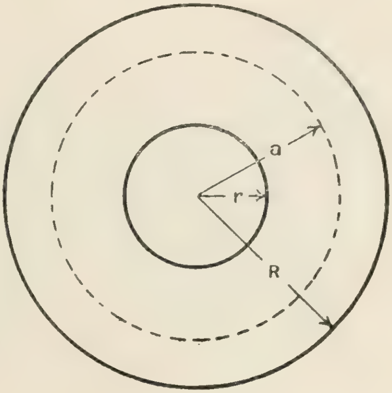

Fig. 67.

The electric stress in the dielectric at a distance a cms. from the centre is, by § 25,

$$F_a=\frac{2q}{\epsilon a}\text{ dynes.}$$

At the surface of the core of the cable the dielectric stress is therefore a maximum, and is given by—

$$F_r=\frac{2q}{\epsilon r}\text{ dynes,}$$

and

$$F_a a=F_r r.$$

Also

$$r=\frac{2q}{F_r\epsilon}=\frac{2VC}{F_r\epsilon},$$

or

$$r=\frac{V}{F_r}\,\frac{1}{\log_e\frac{R}{r}}\text{ cms.}$$

If V is in *volts*, then (§ 57)—

$$F_r=-\frac{dV}{300\,dr}\text{ dynes,}$$

since 1 electrostatic unit of p.d.=300 volts (see § 28).

Hence, when V is in volts,

$$r=-\frac{\frac{V}{300}}{\frac{dV}{300\,dr}}\,\frac{1}{\log_e\frac{R}{r}}\text{ cms.;}$$

that is,
$$r=\frac{V}{U}\frac{1}{\log_e\frac{R}{r}}\text{ cms.,}$$

where U is the stress in the dielectric at the surface of the core of the cable, and is measured in volts drop per cm. along the radial direction.

The quantity U is the maximum stress to which the dielectric is subjected, and the value of U which is to be adopted in the design of a cable is determined from the dielectric strength of the dielectric used divided by a factor of safety. The dielectric strengths of various insulating material are given in § 56.

For a given value of V and U there is one value of r which gives a minimum value for R, the overall radius of the dielectric. It can also be shown that this minimum value of R corresponds very closely to the maximum economy of material in the cable.

To find the condition that R is a minimum it is only necessary to find the condition for which $\frac{dR}{dr}=0$ in the above equation for r. This condition is that

$$r=\frac{V}{U},$$

that is
$$\log_e\frac{R}{r}=1,$$

or
$$R=er,$$

where e is the base of Naperian logarithms—viz., 2·72.

Example.—Suppose the dielectric of the cable is paper for which the permissible value of U is 60,000 volts per cm., and let the cable be designed for a pressure of 85,000 volts.

The radius of the core for the most economical cable will be—

$$r=\frac{V}{U}=1{\cdot}4\text{ cms.,}$$

and the outside radius of the dielectric will therefore be—

$$R=2{\cdot}72\times1{\cdot}4=3{\cdot}8\text{ cms.}$$

The stress U_a at any point in the dielectric distant a cms. from the centre will be given by—

$$U_a\,a=U\,r,$$

or
$$U_a=\frac{a}{85000}.$$

In Fig. 68 the values of U_a are plotted for the whole thickness of the dielectric.

For high voltage cables the value of r determined by the above method would in general give a core of section much larger than would be required from the point of view of current carrying capacity. The conductor could therefore be made hollow, and may be built up of a number of strands of copper wire arranged in a tubular form. In order to protect the dielectric from the excessive local strains due to the small radius of the individual strands a smooth lead sheath may be drawn over the whole of the strands.

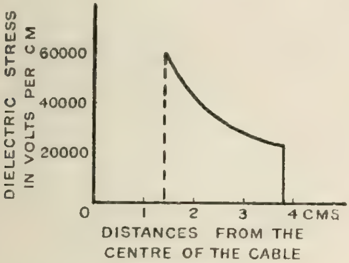

Fig. 68.

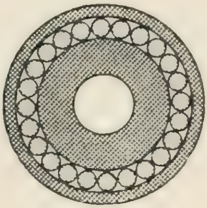

Fig. 69.

In Fig. 69* is shown one such arrangement which also shows an inner lead tube forming a firm support for the stranded conductor.

59. Grading the Dielectric in High Tension Cables.—Reference to the stress diagram of Fig. 68 shows that whereas the dielectric is subject to the maximum stress at the surface of the core, the stress at the outer layers is much less than the maximum allowable, or, in other words, the outer layers of the dielectric are not economically used. If, however, by some means the stress can be so distributed that the value in the outer layers is increased without increasing the stress near the core, a less thickness of insulation will be required for a given cable voltage. This may be done in practice in one of two ways—viz.:

(1) By inserting at some intermediate layer of the dielectric a cylindrical sheath concentric with the core and extending the whole length of the cable. The potential of this inter-sheath is then fixed relatively to the potential of the core by connecting it, for example, to some tapping on the supply transformer winding.

(2) By forming the dielectric in layers of insulating materials having different values of the specific inductive capacities.

Cables in which the dielectric stress is so adjusted are known as **graded cables.**

* See C. J. Beaver, "Cables," *Journal of Inst. of Elec. Eng.*, 1915, vol. liii.

*Method I.—Graded cables in which a metal intersheath is encased in the dielectric.**

In Fig. 70 let the radius of the core of the cable be r cms., the radius of the intersheath ρ cms., and the radius of the outside of the insulation be R cms.

If V and $V\rho$ be the potential of the core and intersheath respectively with reference to the outer armouring (*i.e.*, to earth) it is easy to show that—

$$r = \frac{V - V\rho}{U} \cdot \frac{1}{\log_e \frac{\rho}{r}}$$

and

$$\rho = \frac{V\rho}{U} \cdot \frac{1}{\log_e \frac{R}{\rho}},$$

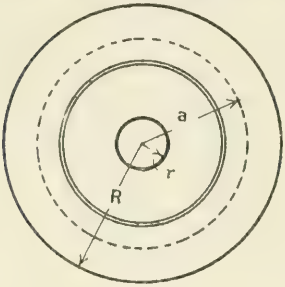

FIG. 70.

in which the same value of the maximum dielectric stress U is assumed for the surface of the core and the outer surface of the intersheath.

Hence the cable pressure is—

$$V = U\left[r \log_e \frac{\rho}{r} + \rho \log_e \frac{R}{\rho}\right] \text{ volts.}$$

For the minimum value of R when the values of V and U are given, it can be shown that—

$$\log_e \frac{R}{\rho} = 1 - \frac{r}{\rho}$$

and

$$\log_e \frac{\rho}{r} = 1,$$

or, by substitution in the equation for V, and calling $\frac{V}{U} = K$,

then

$$\rho = K$$

and

$$\log_e r = \log_e K - 1,$$

$$\log_e R = \log_e K + 1 - \frac{r}{K}.$$

Example.—Assuming as in the example for the ungraded cable (Fig. 68) that

$$V = 85000 \text{ volts}: \quad U = 60000 \text{ volts},$$

then $r = 0{\cdot}52$ cms.: $\rho = 1{\cdot}42$ cms.: $R = 2{\cdot}68$ cms.

It is thus seen that the value of ρ so obtained will fix the intersheath not quite midway between the core and the outer skin of the dielectric, but somewhat nearer the core.

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* C. J. Beaver, "Cables," *Journal of Inst. of Elec. Eng.*, vol. liii.; December, 1914, and March, 1915.

In practice, however, it would be more convenient to place the intersheath centrally in the dielectric. In this case the sheath is made 1 mm. thick and $r=0{\cdot}5$ cm.: $R=2{\cdot}8$ cms. For these conditions the inner radius of the sheath becomes $\rho_1=1{\cdot}6$ cm., and the outer radius of the sheath will be $\rho_2=1{\cdot}7$ cm., and $R=2{\cdot}8$ cms.

In Fig. 71 is shown the stress diagrams for these conditions.

It will be seen by comparing this cable with that shown in Fig. 68 (which is designed for the same electrical requirements) that the provision of the intersheath results in a marked economy of material.

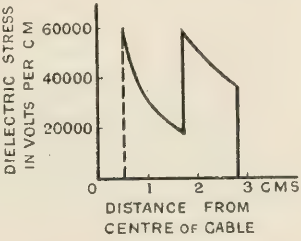

FIG. 71.

The potential difference between the cable and intersheath necessary to satisfy the conditions of Fig. 71 is—

$$V - V\rho = 35000 \text{ volts},$$

and the p.d. between the intersheath and the outer armouring is—

$$V\rho = 50000 \text{ volts}.$$

The voltage of the intersheath must be maintained at this value by connecting it to a suitable tapping point on the supply generator or transformer.

*Method II.—Graded cables in which the dielectric is formed of layers of insulation of different specific inductive capacities.**

Let the radius of the core of the cable be r cms., and let ρ_1 cms. be the outer radius of the first layer of insulation, ρ_2 cms. the outer radius of the second layer of insulation, and so on. If ϵ_1, ϵ_2 . . . be the respective specific inductive capacities of the layers of insulation, reference to § 58 will show that the following relationships will hold—viz.:

$$F_{1\,max}\,\epsilon_1 r = F_{\rho 1}\epsilon_1\rho_1 = F_{2\,max}\,\epsilon_2\rho_1,$$

and so on for subsequent layers.

In the above equations

$F_{1\,max}$ is the maximum stress in dielectric 1.
$F_{2\,max}$,, ,, ,, 2.

and $F_{\rho 1}$ is the stress at the outer skin of dielectric 1.

If there are two different layers of dielectric and the maximum stress in each is taken to be the same, and if

$$F_{1\,max} = F_{2\,max},$$

then

$$\frac{r}{\rho_1} = \frac{\epsilon_2}{\epsilon_1},$$

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* M. O'Gorman, *Journal of Inst. of Elec. Eng.*, 1901, vol xxx.

and hence the layer of insulation near the dielectric must have a higher value of ϵ than the next layer.

Objections have been raised to this method of grading the dielectric on account of the possible chemical action between the adjacent surfaces of the different insulating material.

C. J. Beaver* has suggested the following summary of the conditions under which grading the dielectric is commercially advantageous—viz.:

1. Working pressures exceeding about 70,000 volts.
2. Maximum dielectric stresses above 85,000 volts per cm.
3. Where the cable diameter would exceed about 3 inches.

The foregoing discussion applies more particularly to the case of alternating current cables.

For direct current cables the stress distribution in the dielectric when graded according to Method II. depends when steady conditions have been reached upon the *insulation resistance* of the individual insulating layers, and only at the instant of switching the cable on to the direct current supply is the stress distribution dependent upon the action of the cable and dielectric as a condenser.

60. Corona Effects.—When the potential difference between the conductors in air reaches a high value the conductors become luminous, and if the potential difference is further increased brush discharge sets in accompanied by a hissing noise. Still further increase of potential difference causes a spark or arc to pass between the conductors (see also § 27).

The potential difference at which brush discharge occurs between two parallel cylindrical conductors in air is given by Ryan as—

$$\frac{17{\cdot}94\,b}{459{\cdot}2+t}\,138000\,(r+0{\cdot}18)\log_{10}\frac{a}{r}\text{ volts,}$$

where b inches is the barometric height,
,, t° F. is the temperature,
,, r cms. is the radius of the conductor,
,, a cms. is the distance between the conductors.

This formula only applies for conductors of $\frac{1}{4}$ inch diameter and upwards.

Brush discharge involves a loss of power, and as the voltage is increased beyond the critical value at which brush discharge takes place, the loss of power increases enormously.

Mershon has found that when the voltage is near the critical value at which brush discharge takes place, the loss of power depends very considerably on the humidity of the atmosphere and on the presence of floating particles in the air.

61. Conditions of Equilibrium of the Tubes of Strain of an Electric Field.—It has been shown (§ 23) that the intensity F of the electric force close to the surface of a charged conductor is $\frac{4\pi\sigma}{\epsilon}$, where σ is the surface density of the charge. It has also been shown that the force acting normally outwards on each sq. cm. of the surface of the conductor is (§ 27)—

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* *Loc. cit.*

$$\frac{2\pi\sigma^2}{\epsilon} \text{ dynes.}$$

Consider now the tubes of strain starting from unit area of the charged surface (see § 55). By definition, σ unit tubes of strain start from each unit area of the positively charged surface. The force on the unit surface is thus the same as if *each tube* exerted a pull equal to $\frac{1}{2}$ F—*i.e.*, $\frac{2\pi\sigma}{\epsilon}$ dynes, the pull of the σ tubes per unit area then being $\frac{2\pi\sigma^2}{\epsilon}$ dynes.

Hence the pull on the surface of the charged conductor is the same as if the tubes were in a state of tension, the tension at any point being equal to one-half the value of the electric intensity at that point.

The tension across each sq. cm. of area is thus:

$$\frac{2\pi\sigma^2}{\epsilon} \text{ dynes} = \frac{2\pi}{\epsilon}\frac{\epsilon^2 F^2}{16\pi^2} \text{ dynes} = \epsilon\frac{F^2}{8\pi} \text{ dynes.}$$

This tension at the ends of the tubes will not, however, maintain the dielectric in equilibrium, unless the field is uniform—that is, unless the lines of force are straight and parallel to each other.

If, however, in addition to this tension along the axis of the tubes there is a pressure acting at right angles to the sides of the tubes and equal to $\frac{\epsilon F^2}{8\pi}$ dynes per sq. cm., the tubes will be in equilibrium.

This can easily be seen to be true for some simple cases.

Consider a charge Q electrostatic c.g.s. units isolated at the point O. The lines of force will be straight lines radiating uniformly from the point outwards. Draw the two hemispherical shells OABC and DEF (Fig. 72), the radii being respectively r and $r+\delta r$ cms.

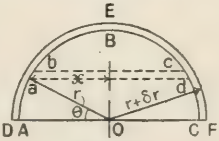

FIG. 72.

Consider the effect of the pull normally inwards on the inner surface of the hemisphere. The intensity of this pull is—

$$\epsilon\frac{F^2}{8\pi} \text{ dynes per sq. cm.}$$

But $$F=\frac{Q}{\epsilon r^2}\text{ dynes.}$$

Hence the intensity of the pull is—

$$\frac{Q^2}{8\pi r^4\epsilon}\text{ dynes per sq. cm.}$$

For the slice *abcd* formed by two horizontal sections of the hemisphere the total inward pull will be—

$$\frac{Q^2}{8\pi r^4\epsilon}2\pi xr\delta\theta\text{ dynes.}$$

But $x=r\cos\theta$, hence the inward pull on the slice is—

$$=\frac{Q^2}{8\pi r^4\epsilon}2\pi r^2\cos\theta\delta\theta\text{ dynes}$$

$$=\frac{Q^2}{4r^2\epsilon}\cos\theta\delta\theta\text{ dynes.}$$

The component vertically downwards will be—

$$\frac{Q^2}{4r^2\epsilon}\cos\theta\sin\theta\delta\theta\text{ dynes,}$$

and the horizontal component will be zero.

Hence the resultant effect of the radial pull inside the hemisphere will be a vertical inward pull.

The total vertical inward pull on the inside of the hemisphere ABC will therefore be—

$$\int_0^{\frac{\pi}{2}}\frac{Q^2}{4r^2\epsilon}\,\frac{\sin 2\theta}{2}\,d\theta\text{ dynes}$$

$$=\frac{Q^2}{8r^2\epsilon}\text{ dynes.}$$

Similarly, for the outer surface of the hemisphere there will be a total vertical outward pull—

$$=\frac{Q^2}{8(r+\delta r)^2\epsilon}\text{ dynes.}$$

Hence for the hemispherical shell ODEF, the total resultant of the pulls inward and outward will be a vertically inward pull—

$$=\frac{Q^2}{8r^2\epsilon}-\frac{Q^2}{8(r+\delta r)^2\epsilon}\text{ dynes}$$

$$=\frac{Q^2\delta r}{4\epsilon r^3}\text{ dynes.}$$

In order to maintain equilibrium, an upward pressure of intensity P must be applied to the rim of the hemispherical shell, such that—

$$P 2\pi r \delta r = \frac{Q^2 \delta r}{4\epsilon r^3},$$

or

$$P = \frac{Q^2}{8\pi r^4 \epsilon} \text{ dynes per sq. cm.}$$

Thus a pressure normal to the tubes of strain and of intensity equal to that of the tension along the tubes will maintain equilibrium.

Instead of considering the whole hemispherical shell, suppose a small approximately plane element ABCD of radial thickness 2δ cms. be considered as shown in Fig. 73:

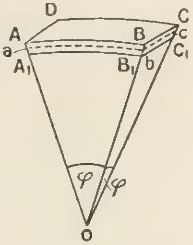

FIG. 73.

Let $Oa = Ob = R$:
Angle $AOB =$ Angle $BOC = \phi$:
Area $ABCD = (R+\delta)^2\phi^2$:
Area $A_1B_1C_1D_1 = (R-\delta)^2\phi^2$.

The pull on area ABCD

$$= \text{Area } ABCD \times \frac{F^2\epsilon}{8\pi} = \frac{(R+\delta)^2\phi^2}{8\pi\epsilon}\left[\frac{Q}{(R+\delta)^2}\right]^2 \text{ dynes outwards.}$$

The pull on area $A_1B_1C_1D_1$

$$= \frac{(R-\delta)^2\phi^2}{8\pi\epsilon}\left[\frac{Q}{(R-\delta)^2}\right]^2 \text{ dynes inwards.}$$

Hence the total pull

$$= \frac{Q^2\phi^2}{8\pi\epsilon}\left[\frac{1}{(R-\delta)^2} - \frac{1}{(R+\delta)^2}\right] \text{ dynes}$$

$$= \frac{Q^2\phi^2}{8\pi\epsilon} \frac{4R\delta}{R^4} \text{ dynes}$$

$$= \frac{Q^2\phi^2\delta}{\epsilon 2\pi R^3} \text{ dynes inwards.}$$

If P is the pressure on the edges of the element, the total pressure outwards is—

$$= 4P2R\phi\delta\frac{\phi}{2} \text{ dynes,}$$

$$= 4PR\phi^2\delta \text{ dynes,}$$

since the area of each edge is $2R\phi\delta$.

Hence for equilibrium the total outward pressure $4PR\phi^2\delta$ dynes due to the pressure on the edges must be equal to the total inward pull—

$$\frac{Q^2\phi^2\delta}{\epsilon 2\pi R^3}\text{ dynes}$$

due to the tension on the faces of the element, or

$$4PR\phi^2\delta=\frac{Q^2\phi^2\delta}{\epsilon 2\pi R^3},$$

i.e.,

$$P=\frac{Q^2}{\epsilon 8\pi R^4}$$

$$=\frac{\epsilon F^2}{8\pi}\text{ dynes.}$$

Hence if the intensity of the pressure normal to the tubes of strain is equal to the intensity of the pull along the tubes of strain equilibrium will be established.

It can be shown that this result holds generally whatever the shape of the tubes of force may be.

62. Refraction of the Lines of Force on passing from One Insulating Medium to Another.—Consider a tube of strain passing from one insulating medium M to another N, and let XY be the boundary of the two media (Fig. 74).

A tube of strain must begin on a positive charge, and end on an equal negative charge.

Let D_1 be the strain in the tube in medium M (§ 55), and let D_2 be the strain in the tube in medium N.

The product: strain × cross-section of the tube, must be the same for both media (§ 55).

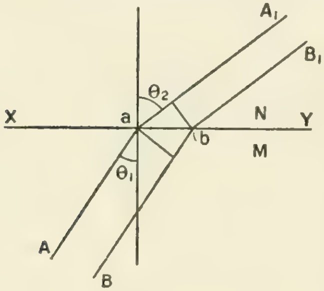

Fig. 74.

Let α be the area *ab* of the boundary surface intercepted by the tube.

Then $D_1\alpha\cos\theta_1=D_2\alpha\cos\theta_2$, since this equation expresses that the tube will end on equal quantities of electricity whatever the nature of the insulating media through which it may pass (§ 55).

Moreover, the tangential intensity on *ab* must be the same for each medium—that is, if F_1 be the electric intensity in medium M, and F_2 be the intensity in medium N, then $F_1\sin\theta_1=F_2\sin\theta_2$, or the intensity of the force parallel to the boundary surface *ab* is the same at two points indefinitely near together, and one on each side of the boundary. That this is so may be seen as follows:

In Fig. 75 let AB be the boundary surface of two different media. Consider the forces at points P and Q respectively, and in the direction parallel to AB, the points P and Q being taken indefinitely close to the boundary AB.

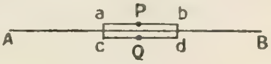

FIG. 75.

Draw a rectangle *abcd* of which the sides *ab* and *cd* are parallel to AB, and pass through the points P and Q respectively, and the length of *ab* and *cd* is such that the intensity of the force in the direction parallel to AB may be taken as constant along each line *ab* and *cd*.

Suppose, now, a small charge be carried round the rectangle *abdc*. The work done in performing this cycle must be zero, otherwise the energy of the system would change without changing the distribution of the charges. Further, the work done in moving along the indefinitely short sides *bd* and *ca* will be negligibly small. Hence the work done in moving along the side *ab* will be equal and opposite to that done in moving along the side *dc*—that is to say, the intensity of the force along *ab* is the same as the intensity of the force along *cd*.

From the above equations it follows that

$$\frac{F_1}{D_1}\tan\theta_1=\frac{F_2}{D_2}\tan\theta_2.$$

But $$D_1=\frac{\epsilon_1 F_1}{4\pi}:\ D_2=\frac{\epsilon_2 F_2}{4\pi}\ (\S\ 55).$$

Hence $$\frac{F_1}{D_1}=\frac{4\pi}{\epsilon_1}:\ \frac{F_2}{D_2}=\frac{4\pi}{\epsilon_2}.$$

Therefore $$\frac{4\pi}{\epsilon_1}\tan\theta_1=\frac{4\pi}{\epsilon_2}\tan\theta_2,$$

or $$\frac{\tan\theta_1}{\tan\theta_2}=\frac{\epsilon_1}{\epsilon_2}.$$

Suppose medium N is air—*i.e.*, $\epsilon_2=1$—and medium M is a conductor—*i.e.*, $\epsilon_1=\infty$,

Then $$\tan\theta_2=\frac{1}{\infty}\tan\theta_1,$$

or $$\tan\theta_2=0.$$

Therefore $$\theta_2=0.$$

Hence the lines of force in air are perpendicular to the surface of a conductor.

In Fig. 76 are shown the lines of force for the case in which a sphere of paraffin is placed in an originally uniform electric field. The value of the specific inductive capacity for paraffin is 2·3.

By § 54 the energy per c. cm. is—

$$\frac{\epsilon F^2}{8\pi} \text{ ergs}$$

$$= \frac{2\pi D^2}{\epsilon} \text{ ergs,}$$

where D is the electric strain.

When a given number of tubes of strain pass through the paraffin, the energy is less than when they pass through the air, and since, according to a well-known principle, the energy tends to become a minimum the lines of force crowd into the paraffin.

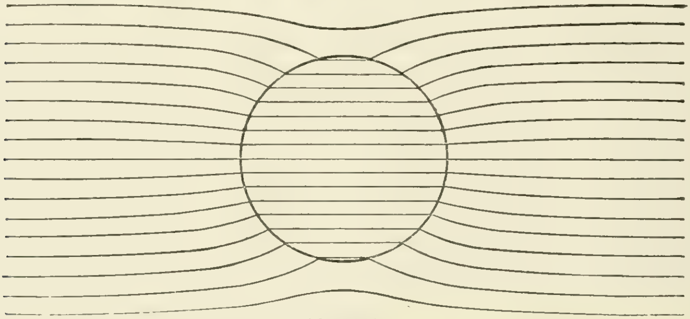

Fig. 76.

In Fig. 77 is shown the lines of force for the case in which a metal sphere is placed in an originally uniform electric field.

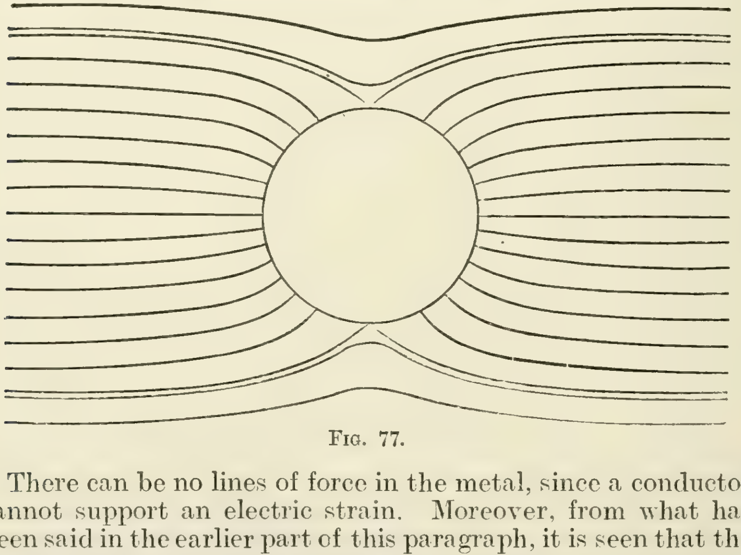

Fig. 77.

There can be no lines of force in the metal, since a conductor cannot support an electric strain. Moreover, from what has been said in the earlier part of this paragraph, it is seen that the lines of force must meet the surface of the sphere at right angles to the surface. (Figs. 76 and 77 are from Lord Kelvin's Papers).

CHAPTER V

ELECTRICAL MACHINES AND ELECTROMETERS

63. Electrical Machine.—An electrical machine is an apparatus by means of which mechanical energy is transformed into electric energy. The mechanical energy is utilised in transferring a body from one place where it has received an electric charge, to another place where it gives its charge up to a second body, the mechanical energy being expended in overcoming the electric forces which oppose the transfer of the charged body.

Electrical machines are of two general types—viz.: (1) Frictional machines, in which the electrification of the transferred body is produced by friction—*e.g.*, the plate-glass frictional machine; and (2) induction machines, in which the electrification of the transferred body is produced by induction.

Of these two types of electrical machines, the *induction type* is the most commonly used.

The efficiency of the energy transformation of electrical machines is extremely low. Of the total mechanical energy supplied the proportion required for overcoming the friction of the moving parts is enormously greater than that actually utilised in overcoming the opposing electrical forces.

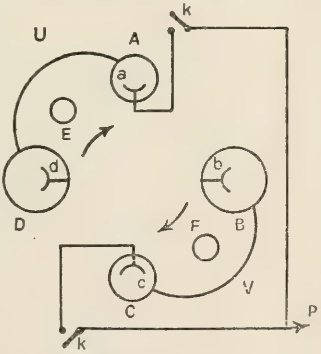

FIG. 78.

The general action of an induction type of electrical machine may be explained by reference to Figs. 78 and 79. Let A, B, C, D represent four spherical conducting shells, each insulated and arranged as shown. Let the shells A and C be precisely similar, and arranged diametrically opposite each other, also let the shells B and D be precisely similar to each other, and placed diametrically opposite. Suppose that the shells A and D are metallically connected to each other, and similarly the shells B and C. It is to be assumed that the inductive action of A and D on B and C is negligible.

Let E and F be two spherical conductors precisely alike, and of a size such that each may be enclosed in any of the four shells A, B, C, D, In Fig. 79 is shown a section of the conducting shell A, this being identical in every respect with the shell C. The support S (Fig. 79) is of insulating material, and is capped with a metal plate *a* on which the conductors E or F may rest. A wire connected to the plate *a* passes through a hole in the insulating support S to the contact keys *kk* (Fig. 78).

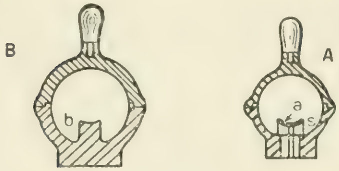

Fig. 79.

Fig. 79 also shows a section of the conducting shell B which is identical in every respect with the shell D. The support *b*, however, is of metal, and cast in one piece with the shell. Each of the conductors E and F may rest on the support *b*, and thus be completely enclosed in the shell.

Let U be the potential of the conductors A and D, and let V be the potential of the conductors B and C, and, initially, let

$$U=U_o,$$
$$V=V_o.$$

Enclose E in A and F in C, and close the contacts *kk*, thus bringing E and F into connection with each other, and with a body at the constant potential P, such as the earth. Then E receives a charge $=a\ (P-U_o)$ where a is a constant. Simultaneously F being enclosed in C receives an induced charge $=a\ (P-V_o)$.

Now open the contacts *kk* and remove the conductors E and F without allowing either to touch the shell, and close up the shells A and C again. Now enclose E in B and F in D. The conductors BC will consequently have their charge augmented by an amount $a\ (P-U_o)$, and hence the potential of BC will rise by an amount

$$\frac{a}{c}\,(P-U_o),$$

where c is the capacity of B and C together.

Hence the potential of B and C is now

$$V_1=V_o+\frac{a}{c}\,(P-U_o),$$

and similarly that of D and A is

$$U_1=U_o+\frac{a}{c}\,(P-V_o).$$

Remove E from B and F from D, and close up the shells B and D again. The first half of a revolution of E and F round the circle on which A, B, C, D are placed is thus completed.

After n half revolutions of E and F round this circle—*i.e.*, after $\frac{n}{2}$ revolutions.

$$V_n = V_{n-1} + r(P - U_{n-1}),$$
$$U_n = U_{n-1} + r(P - V_{n-1}),$$

where $r = \frac{a}{c}$ and is a positive number.

Hence $$V_n - U_n = (V_{n-1} - U_{n-1})(1+r);$$
that is $$V_n - U_n = (V_o - U_o)(1+r)^n.$$

The potential difference between AD and BC rises, therefore, in geometrical progression but maintains the original sign.

The machine thus gives a means of determining the sign of a given potential difference which is too small to be directly measured.

Further, if $(1+r)$ can be taken to be strictly constant, the original potential difference $V_o - U_o$ may be determined from a measurement of the potential difference $V_n - U_n$ which develops after $\frac{n}{2}$ revolutions—viz.:

$$V_o - U_o = \frac{V_n - U_n}{(1+r)^n}$$

64. The Wimshurst Machine.—This induction machine is in practice one of the most reliable and easiest to work.

Two identically similar plates of glass are mounted on a horizontal spindle so as to rotate in opposite directions when the handle is turned. Each glass plate is provided with a number of strips of tinfoil equally spaced and arranged on the surface near the rim. Diametral conductors (called *neutralising rods*) are fixed, one opposite each plate, and making an angle of about 90° with each other. Small brushes are fixed to the ends of each neutralising rod, and graze the tinfoil strips on the plates as the latter rotate.

The rotation of the plates with respect to the neutralising rods is such that a radius which at any instant is horizontal becomes parallel to the corresponding diametral conductor by turning through an acute angle. At opposite ends of the horizontal diameter two ⊏-shaped conductors fitted with points (combs) embrace the plates so that the points come close to the strips of tinfoil. Each comb is connected to an adjustable discharge knob, and to the inner coating of a Leyden jar, the outer coatings of the jars being connected together.

The action of the machine may be explained as follows with reference to Fig. 80: The two glass plates are here represented by two cylinders, the neutralising rods being shown by mm_1

and nn_1, and the collecting combed conductors by cc and c_1c_1. Let the rotation be as shown by the arrows in Fig. 80. Assume that in some way a segment a_1 has become charged with, say, positive electricity. One way of doing this is to bring a negatively charged body, say a stick of rubbed sealing-wax, near the segment a_1 as it passes under the brush m_1. As the inner cylinder turns, the positively electrified segment a_1 comes to the neighbourhood of the brush n_1, and consequently as a segment b_1 passes under the brush n_1 negative electricity is induced on it, and positive electricity repelled to the segment b under the opposite brush n. Hence, when the segments leave the brush n_1 they are negatively electrified, and when they leave the brush n they are positively electrified. Consequently the segments on the outer cylinder from n_1 to c are negatively electrified, and the segments from n to c_1 are positively electrified. On the inner cylinder the segments from m_1 to c_1 are positively electrified, and the segments from m to c are negatively electrified. As the electrified segments pass the combs of the collectors cc_1 they become discharged and the collectors correspondingly charged. Thus, as the electrified segments of the outer cylinder pass the comb of c_1c_1 the positive charge on the segment draws a negative charge from the points of the comb (see § 35, Fig. 38). The positive charge on the collector c_1c_1 is thus increased by an amount corresponding to the charge which was on the segment as it passed under the comb. Similarly for the positively charged segments of the inner cylinder. The negatively charged segments of both cylinders become similarly discharged as they pass the collector cc. The segments on the inner cylinder between c and m_1 and between c_1 and m are practically discharged, as also are the segments on the outer cylinder between c and n and between c_1 and n_1.

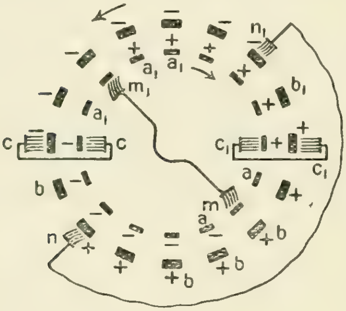

Fig. 80.

It is important to notice that the charge induced on any segment, say the negative charge induced on b_1 as it passes n_1, is greater than the positive charge on each segment a_1, since the charge on b_1 is induced by more than one of the positively charged segments a_1 which are in the neighbourhood of b_1 as it passes under the brush n_1. Consequently the charge induced on each segment becomes increasingly greater as the cylinders revolve, the limit being determined by the leakage over the insulating material of the charged parts. It is further to be noticed that the potential of a segment a_1 is relatively low as it passes the negatively charged segment b_1 under the brush n_1, but when the positively charged segment a_1 reaches the comb c_1 it comes opposite a similarly charged segment b of the outer cylinder, and its potential is very much increased.

The potential difference of the combs cc and c_1c_1 become gradually greater until eventually sparks several inches long may be obtained between the discharge knobs.

The function of the Leyden jar condensers connected respectively to the combed collectors cc and c_1c_1 is to increase the capacity C of these conductors, and so increase the quantity of electricity accumulated before the sparking potential between the knobs is reached. If V is the sparking potential and C the combined capacity of the collectors and Leyden jars, then the quantity of electricity discharged through the spark is given by—

$$Q = VC,$$

and the energy of the spark is—

$$\tfrac{1}{2}QV = \tfrac{1}{2}V^2C.$$

Such a machine is useful for obtaining potential differences of the order of several hundred thousand volts for testing the dielectric strength (see § 56) of materials with a unidirectional p.d. for special purposes.

Wimshurst machines are also sometimes used for X-ray work.

65. The Replenisher.—This is a type of induction machine devised by Lord Kelvin for use with his electrometers (§§ 66 and 67). A diagrammatical sketch of the apparatus is shown in Fig. 81, in which A and B are the main conductors, the potential difference between which it is required to vary, and P and Q are conductors secured to an insulating rod which may be revolved about an axis O concentric with the axis of the main cylindrical-shaped conductors A, B. The action is simple. A small positive charge given to the conductor A induces a negative charge on P as the latter passes the contact finger C, and a positive charge on Q as it passes D. When P touches the contact b and Q touches a the conductors P and Q become discharged, each being in effect connected to the inside of a hollow conductor, and A and B therefore receive corresponding additional charges of positive and negative electricity respectively. Thus, as the conductors PQ are revolved in one direction the potential difference between P and Q is increased, whereas rotation in the opposite direction reduces the potential difference.

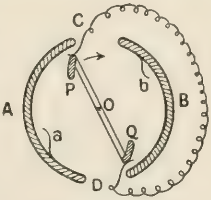

Fig. 81.

66. Absolute (Attracted Disc) Electrometer.—This apparatus is also due to Lord Kelvin. The principle of its action is the attraction which exists between oppositely electrified plates which are arranged parallel to each other at a small distance apart (see §§ 27 and 61).

In Fig. 82 let A and B be two oppositely electrified parallel plates distant d cms. apart of which A is large compared with B, and let A be fixed and B be movable. Let V_A be the potential of A and V_B the potential of B. Neglecting the fringing of the lines of force near the edges of the plate B, the lines of force between the two plates are straight and perpendicular to the planes.

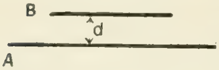

FIG. 82.

Let S be the area of the face of the plate B in square cms., and σ the surface density of the charge in electrostatic c.g.s. units. The intensity of the force of attraction between the plates is—

$$\frac{2\pi\sigma^2}{\epsilon} \text{ dynes per sq. cm. of surface (§ 27).}$$

For the whole surface the total force is—

$$F = \frac{2\pi\sigma^2 S}{\epsilon} \text{ dynes.}$$

But by § 32

$$V_B - V_A = \frac{4\pi\sigma d}{\epsilon} \text{ electrostatic c.g.s. units,}$$

or

$$\frac{\sigma}{\epsilon} = \frac{V_B - V_A}{4\pi d}.$$

Hence $$F = \frac{2\pi\sigma^2 S}{\epsilon} = \frac{S\epsilon}{8\pi d^2}(V_B - V_A)^2 \text{ dynes;}$$

i.e., $$V_B - V_A = d\sqrt{\frac{8\pi F}{S\epsilon}} \text{ electrostatic c.g.s. units,}$$

In order to avoid the disturbance due to the fringing of the lines at the edges of the movable plate B, the following device is used: The plate B is a disc cut out of a larger plate CD (Fig. 83), called the *guard-ring*. The plate B can move freely in the circular hole in CD and is in metallic connection with this plate, and thus is maintained at the same potential as CD. Actually CD is the bottom of a metal box, so that any electrical action which might be exerted from the outside on the upper face of the plate B is avoided. The effective area of the plate B is then to be taken as the mean of the area of the face of the disc B,

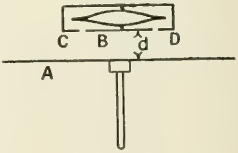

FIG. 83.

and the area of the corresponding hole in the guard-ring. When the electrometer is in use the plate B is kept at a constant potential by means of the *replenisher* (§ 65), and a special device makes it possible to check this potential.

The plate B is supported by a spring, which keeps it slightly raised above the level of the guard-ring when the instrument is discharged.

The method of using the instrument for measuring a given potential difference is as follows:

Suppose it is desired to measure the potential difference $V^1 - V$ between two bodies. The lower (fixed) plate A is connected to the lead at the potential, say V, and the (movable) plate B to the lead at the potential V^1. The plate A is then raised until the plate B, as observed by means of the lenses, appears exactly in the plane of the guard-ring CD.

Let W be the mass in grammes whose weight causes the plate B to sink to the level of the guard-ring when the electrometer is discharged. Then

$$F = Wg.$$

Therefore $$V^1 - V = d\sqrt{\frac{8\pi Wg}{S\epsilon}},$$

where g is the acceleration due to gravity in cms. per second per second ($g = 981$).

In practice the distance d is not easily measured accurately, and hence the following procedure is usually adopted:

The lower plate A is put to earth so that V becomes zero, and the reading of the micrometer taken so that the conditions are satisfied that the plate B is in the plane of the guard-ring CD. Let this distance be d^1 cms., then

$$V_1 = d^1\sqrt{\frac{8\pi Wg}{S\epsilon}},$$

V_1 being the potential at which the plate B is maintained by special arrangements—viz., by means of the replenisher (§ 65).

The plate A is then insulated from earth and put in contact with the body whose potential V it is desired to measure, and the micrometer adjustment again made and read. Let this reading be d cms., then

$$V_1 - V = d\sqrt{\frac{8\pi Wg}{S\epsilon}}.$$

Hence $V = (d^1 - d)\sqrt{\dfrac{8\pi Wg}{S\epsilon}}$ electrostatic c.g.s. units, and it is only necessary to read the two adjustments d and d^1, all the other quantities being constants of the instrument.

In order to keep the plate B at the constant potential, the whole system of plate B, guard-ring and box CD, is connected with the inner coating of a Leyden jar formed by the glass case of the instrument. The replenisher (§ 65) is used to maintain the potential V_1 of B constant, and to verify that this potential is being maintained constant a special device is used which is in effect an auxiliary electrometer.

67. The Quadrant Electrometer.—This type of electrometer was also devised by Lord Kelvin, and is very much more sensitive than the attracted disc type of electrometer. It consists of four hollow quadrants within which a light vane or " needle " suspended by quartz fibres (bifilar suspension) freely moves. The shape of the vane is roughly a figure-of-eight (see Fig. 84), a portion of which may be removed to lighten the vane as much as possible. Each quadrant is supported by an insulating support, and the two diametrically opposite quadrants are electrically connected together (Fig. 84). From each pair of quadrants a connection is brought to a terminal outside the case. From the vane a piece of platinum is suspended by means of a fine wire, and the platinum dips in sulphuric acid contained in a glass jar, the outside of which is coated with tinfoil and earth connected (Fig. 85). The glass jar with sulphuric acid and tinfoil forms a condenser of large capacity, and the vane is thus electrically part of the sulphuric acid boundary of this condenser. The acid also serves to keep the instrument free from moisture.

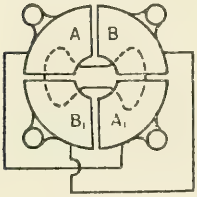

Fig. 84.

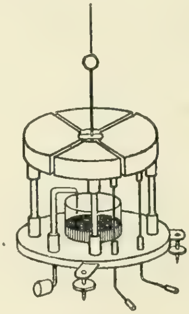

Fig. 85.

In the normal condition of rest the axis of the vane is in line with one pair of the slits, as shown in Fig. 84. The relative dimensions of the parts are such that the radial edges of the vane and the quadrants are far apart, as compared with the breadth of the slits and the depth of the inside of the quadrants.

A mirror is secured to the bifilar suspension, and the deflection of the needle is read by means of a reflected beam of light. The whole instrument is enclosed in a glass case.

To measure small differences of potential the two pairs of quadrants A and B are connected respectively to the points between which the potential difference is to be measured, and the needle C is charged to a high potential.

The deflection θ of the needle is then given by the expression—

$$\theta = K(V_A - V_B)\,[V_C - \tfrac{1}{2}(V_A + V_B)],$$

or if V_C is large compared with V_A and V_B—

$$\theta = KV_C\,(V_A - V_B),$$

where $V_A - V_B$ is the potential difference it is desired to measure—*i.e.*, the potential difference between the opposite pairs of quadrants A and B, and V_C is the potential to which the needle is raised.

In order to measure large differences of potential the needle is connected to one pair of quadrants, say B, so that—

$$V_C = V_A.$$

The deflection of the needle is then—

$$\theta = \tfrac{1}{2}K(V_A - V_B)^2.$$

In this case the separate factor V_C is avoided and the deflection is directly proportional to the square of the potential difference it is desired to measure.

Thus, if the opposite pairs of quadrants are connected to the inner and outer coatings respectively of a Leyden jar, the deflection will be proportional to the energy of the jar.

The constant K is proportional to the specific inductive capacity ϵ, and the instrument may thus be used to measure the value of ϵ for fluids.

Further, suppose a condenser of capacity C_1 be charged with a quantity Q and let V be the *potential difference* between the condenser boundaries.

Now let the condenser share its charge with another condenser of capacity C_2, and let V^1 now be the common *potential difference* between the boundaries of the condensers.

Then
$$\begin{aligned} Q &= C_1V \\ &= (C_1 + C_2)V^1, \end{aligned}$$

or
$$\frac{C_2}{C_1} = \frac{V - V^1}{V^1}.$$

The potential differences V and V^1 can be measured by means of the electrometer, and thus the capacity C_2 compared with the capacity C_1. If C_1 is a standard condenser whose capacity is known, the value of the capacity C_2 is at once found.

This method of comparing the capacity of condensers also gives a means of finding the values of the specific inductive capacities ϵ of different dielectrics.

The formulæ given above for the deflection of the vane of the quadrant electrometer may be deduced as follows:

Let V_A, V_B, V_C be respectively the potentials of the quadrant pairs A and B and of the vane, and let Q_A, Q_B, Q_C be the corresponding charges.

The lines of force from the vane will extend partly to the quadrant pair A, partly to the quadrant pair B, and partly to other conductors in the neighbourhood of the instrument.

The following equations will then hold—viz.:

$$Q_A = C_{AB}(V_A - V_B) + C_{AC}(V_A - V_C) + C_A V_A,$$
$$Q_B = C_{BA}(V_B - V_A) + C_{BC}(V_B - V_C) + C_B V_B,$$
$$Q_C = C_{CA}(V_C - V_A) + C_{CB}(V_C - V_B) + C_C V_C,$$

where C_{AB} is the capacity of the field between the quadrant pairs A and B, C_{AC} is the capacity of the field between the quadrant pair A and the vane, and C_A is the capacity of the field between the quadrant pair A and the other conductors in the neighbourhood of the instrument.

Similarly for the other constants—

$$C_{BA},\ C_{BC},\ C_B,\ C_{CA},\ C_{CB},\ \text{and}\ C_C.$$

For these constants the following relationships hold,

$$C_{AB} = C_{BA}:\ C_{AC} = C_{CA}:\ C_{BC} = C_{CB}.$$

The total energy of the field will be (see § 52)—

$$W = \tfrac{1}{2}Q_A V_A + \tfrac{1}{2}Q_B V_B + \tfrac{1}{2}Q_C V_C$$
$$i.e.,\ W = \tfrac{1}{2}C_{AB}(V_A - V_B)^2 + \tfrac{1}{2}C_{AC}(V_A - V_C)^2 + \tfrac{1}{2}C_{BC}(V_B - V_C)^2 + \tfrac{1}{2}C_A V_A^2 + \tfrac{1}{2}C_B V_B^2 + \tfrac{1}{2}C_C V_C^2.$$

Now the potentials V_A, V_B, V_C will be independent of the movement of the vane.

Let Θ be the torsional couple due to the suspension wire, and let $\delta\theta$ be the angle through which the vane turns.

Then $\Theta\delta\theta$ will be the work done in turning the vane through the angle $\delta\theta$, but, as was shown in § 53, the work done is equal to the increase of the energy of the electric field.

Hence $$\Theta\delta\theta = \delta W,$$

or $$\Theta = \frac{dW}{d\theta}.$$

Hence

$$\Theta = \tfrac{1}{2}\left[(V_A - V_B)^2 \frac{dC_{AB}}{d\theta} + (V_A - V_C)^2 \frac{dC_{AC}}{d\theta} + (V_B - V_C)^2 \frac{dC_{BC}}{d\theta} + V_A^2 \frac{dC_A}{d\theta} + V_B^2 \frac{dC_B}{d\theta} + V_C^2 \frac{dC_C}{d\theta}\right].$$

As the vane moves from one pair of quadrants, say A, to the other pair B, the diminution of the capacity C_{CA} will be equal to the increase of the capacity C_{CB}. Further, the change in the capacities C_{CA} and C_{CB} will be proportional to the angle turned through, and this will be true, notwithstanding the fringing effect, since the fringing will be constant for small movements of the vane, provided the distance of the radial edges of the vane and quadrants is great compared with the internal depth of the quadrants and the width of the slits between the quadrants.

Suppose the vane moves from the quadrant pair A towards the quadrant pair B. Then

$$\frac{dC_{BC}}{d\theta}=k:\ \frac{dC_{AC}}{d\theta}=-k,$$

where k is a constant.

Also $$\frac{dC_{AB}}{d\theta}=0:\ \frac{dC_{A}}{d\theta}=0:\ \frac{dC_{B}}{d\theta}=0:\ \frac{dC_{C}}{d\theta}=0.$$

Therefore $$\Theta=\tfrac{1}{2}[(V_C-V_B)^2k-(V_C-V_A)^2k],$$

or $$\Theta=\tfrac{1}{2}k[(V_C-V_B)^2-(V_C-V_A)^2].$$

If V_C-V_B is greater than V_C-V_A, the turning moment is positive—that is to say, the vane moves towards the pair of quadrants of which the potential differs most from the potential of the vane.

The expression for the turning moment

$$\Theta=\tfrac{1}{2}k[(V_C-V_B)^2-(V_C-V_A)^2]$$

may be written—

$$\begin{aligned}\Theta&=k[V_C(V_A-V_B)-\tfrac{1}{2}(V_A{}^2-V_B{}^2)]\\&=k(V_A-V_B)[V_C-\tfrac{1}{2}(V_A+V_B)];\end{aligned}$$

or, if V_C is large compared with V_A and V_B,

$$\Theta=k(V_A-V_B)V_C.$$

But the turning moment Θ is proportional to the deflection θ.

Therefore $$\Theta=a\theta,$$

where a is a constant; that is,

$$\theta=K(V_A-V_B)V_C.$$

68. Dolezalek Electrometer.—A modified form of the quadrant electrometer has been introduced by Dolezalek, and on account of its greater sensitiveness and other advantages is much used (Fig. 86).

The needle and quadrants are small, so that the electrostatic capacity of the instrument is small. The quadrants are mounted on ambroid (a composition of amber) pillars, and are thus extremely well insulated. The needle is of paper covered with a metallic film, and is suspended by a quartz fibre of diameter from 0·003 mm. to 0·009 mm. and about 60 mm. long. The moving system is thus made extremely light and sensitive.

The Leyden jar is dispensed with, and the needle is usually charged to a potential of from 50 to 100 volts by means of the terminal K.

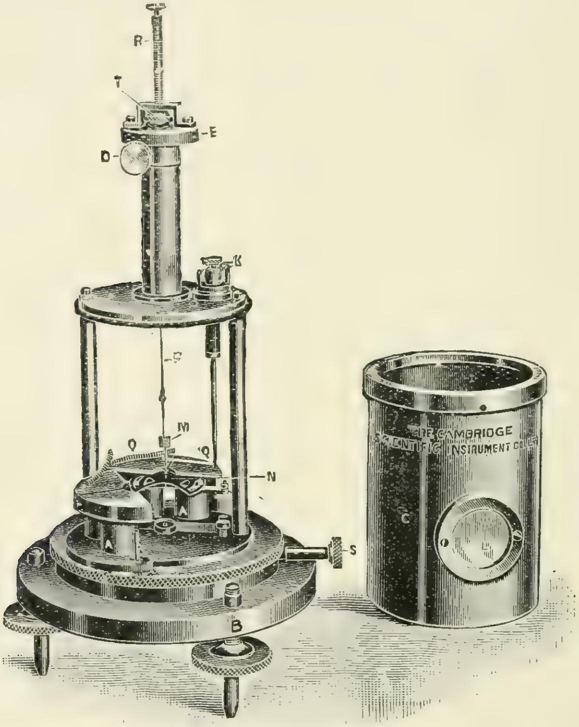

Fig. 86.

As an example of the sensitivity of this instrument the following data will serve:

The quartz fibre suspension was of diameter 0·006 mm. and 60 mm. long. The needle was charged so that the potential difference between the needle and quadrants was 100 volts, the quadrants being earthed whilst the needle was being charged. The needle was then isolated from the supply, and a potential difference of 0·1 volt was established between the two pairs of quadrants. The deflection of the needle was such that the spot of light reflected from the mirror secured to the suspension of the needle moved 150 mm. division of a scale placed at a distance of 1 metre. The instrument was practically "dead beat" (see § 211), and took about thirty seconds to attain the deflection.

SECTION II

MAGNETISM

CHAPTER VI

FIRST PRINCIPLES

69. Natural Magnet.—A mineral called magnetite (of chemical composition Fe_3O_4) has the property of attracting small particles of iron, and was found by the ancients near Magnesia (Lydia), from which locality the name *magnet* is probably derived. A piece of this mineral, exhibiting the property of attracting iron, is called a "natural magnet," as distinguished from steel magnets which are made artificially.

If a natural magnet be rolled in iron filings, it is usually found that the filings concentrate chiefly at two opposite ends of the magnet, and if suspended by a thread, so that the line joining these two ends is horizontal, the magnet will point roughly North and South. This fact was also known to the ancients, and the name "lodestone" (*i.e.*, leading-stone) was also given to the mineral by reason of this directive property.

70. Steel Magnet.—If a bar of steel is stroked in one direction with one end of a piece of lodestone, the steel bar becomes itself a magnet, or is said to be *magnetised*, acquiring properties similar to those of the lodestone—viz., iron filings are attracted by its ends, and it sets roughly North and South when freely suspended, so that the line joining its ends is horizontal.

71. Magnetic Poles.—The two ends of a magnet over which iron filings collect are termed the poles of the magnet, and the end which, when the magnet is freely suspended horizontally, points towards the North is called the North-seeking or North Pole of the magnet, and the end which points towards the South is called the South-seeking or South Pole. The poles are usually denoted by the letters N and S respectively.

72. Position of the Poles.—If a bar of steel is magnetised by stroking one way (*i.e.*, not backwards and forwards) in the direction of its length, and then dipped in iron filings, it is observed, as already stated, that the filings cluster in regions near each end, and hardly any cling to parts near the middle of the bar. The *regions* to which the filings chiefly adhere are termed the poles of the magnet, and it is to be observed that the poles are not definite points but regions (see also § 81).

Magnets are not necessarily bar-shaped, but may be of any shape—*e.g.*, spherical magnets. In all cases however, two regions, or poles, are found to which filings specially adhere.

73. Mutual Action between Magnetic Poles.—Suppose there are two magnets A and B (Fig. 87), and suppose the magnet A is freely suspended on a pivot—as a compass needle, in fact. If, now, the magnet B be brought near the suspended magnet A the following facts will be observed—viz., if the N pole of B be brought near the N pole of A, the magnet A swings round, so that its S pole is as near as possible to the N pole of B. If the S pole of B be brought near the S pole of A, the magnet A will swing round so that its N pole is as near as possible to the S pole of B, or stated generally:

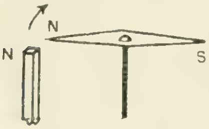

Fig. 87.

Like magnetic poles repel each other,

and

Unlike magnetic poles attract each other.

An inconsistency of nomenclature is to be observed here—viz., the fact that a bar magnet freely suspended horizontally tends to set pointing N and S suggests that the earth itself is a magnet, of which the poles are in the neighbourhood of the geographical N and S poles. If, therefore, the N magnetic pole of the earth is the one in the neighbourhood of the geographical N pole, the pole of a freely suspended horizontal magnet which points towards the N Pole of the earth should be termed the S pole. However, the convention is universally adopted of calling the N-seeking pole of a magnet its N pole and the S-seeking pole its S Pole (see Chapter IX.).

Compass adjusters speak of the N pole of a magnet as the red end, or pole, of the magnet and of the S pole as the blue end. The N pole is painted red and the S pole blue. The magnetism of the northern hemisphere is then said to be of the blue variety and of the southern hemisphere of the red variety.

74. Magnetism.—In § 114 some account is given of the generally accepted theory which explains the phenomena of the magnetisation of a magnet.

For many purposes, however, it is convenient to assume that the magnetic effects of a magnet are due to a substance called **magnetism** distributed over the regions marked out by the filings (§ 72)—*i.e.*, the poles. It is then necessary to distinguish between two kinds of magnetism—viz., *positive magnetism* distributed over the N pole, and *negative magnetism*, distributed over the S pole.

75. A Single Isolated Magnetic Pole cannot be Obtained.—If a bar magnet is broken near the middle, a N pole is not thereby isolated from a S pole, but at the two ends formed by the fracture poles are developed, so that each piece becomes a perfect magnet—viz., the half at the end of which the N pole of the original magnet was situated develops a S pole at the new end formed by the fracture, and the half at the end of which the S pole of the original magnet was situated develops a N pole at the new end formed by the fracture (Fig. 88).

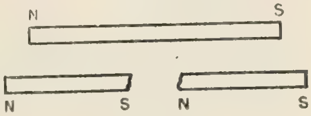

FIG. 88.

If each half be broken into two parts, new poles appear at the fracture, and however often the process of subdivision is repeated the same effect is observed—viz., each part becomes a complete magnet. It is thus found to be impossible to obtain an isolated N or S pole in practice. For experimental purposes the nearest approach to an isolated pole is a very long thin magnet, in which case the N pole is so far removed from the S pole that there is little influence of one pole in the neighbourhood of the other (see § 84).

In the case of static electricity it was found (§ 1) that equal and opposite charges are always simultaneously in existence, but not necessarily on one and the same body.

76. The Poles of a Magnet are of Equal Strength, or the Total Quantity of Magnetism of a Magnet is always Zero.—This is proved by the fact that a magnetised steel bar is acted on by the earth's magnetic field in a purely directive manner—that is to say, there is no resultant force tending to move the magnet bodily. There is no resultant horizontal component, since a magnet when floated on a cork tends merely to set itself in the direction of the N and S poles of the earth, but does not tend to move bodily. There is no resultant vertical component, since a steel bar weighs no more when magnetised than it does when unmagnetised.

Hence the forces acting on a magnetised bar due to the earth's magnetic field form a couple, and the following statement is true:

> **The amount of positive magnetism of a magnet is equal to the amount of negative magnetism, or otherwise stated, the total magnetism of a magnet is zero.**

77. Lines of Force.—If a curve be drawn in a magnetic field (§ 78), such that the direction of the curve at every point is the direction of the magnetic force at that point, such a curve is termed a line of force. The direction of the line of force at any place is the direction in which a N pole at that place would tend to move (see also § 86).

78. The Magnetic Field.—The field of a magnet is that region throughout which the influence of the magnet may be detected.

A very striking demonstration of the magnetic field of a magnet is obtained by means of iron filings, as follows: Take two bar magnets of the same size, set them up vertical and parallel, and place a piece of cardboard over the upper ends of the magnets. Now sprinkle finely-sifted iron filings over the cardboard, at the same time gently tapping the cardboard. It will then be found that the filings arrange themselves in well-defined curves, as shown in Figs. 89 and 90.

In Fig. 89 the two magnets are arranged with their unlike poles adjacent, and in Fig. 90 with their like poles adjacent.

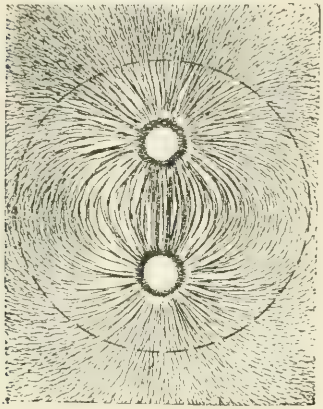

Fig. 89.

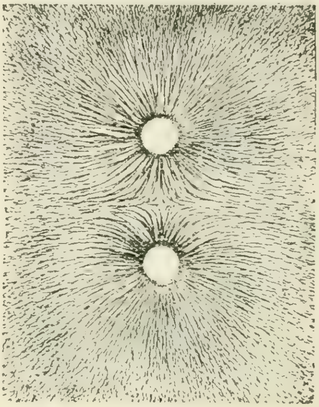

Fig. 90.

If only one magnet is used the general arrangement of the filings will be along radiating straight lines as indicated in Fig. 91.

Another method of demonstrating the shape of the lines of force is by means of a small compass needle. If the compass needle is moved from one pole to the other, following the direction in which it sets at every new position of the needle, the path of the needle will give the direction of the line of force, and each journey of the needle may be made along a different line of force.

Generally speaking, the field of a magnet is only very slightly affected by the nature of the medium, unless the medium is a magnetic substance. Thus, instead of using cardboard for obtaining the pictures of the magnetic fields shown in Figs. 89, 90, 91, glass, zinc, or other "non-magnetic" substance may be used. In this connection reference should be made to §§ 120 and 121.

79. Earth's Field.—The region about the earth is a magnetic field, since a magnet is there subject to forces tending to make it set in a definite direction.

If a magnetised bar is freely suspended horizontally—*e.g.*, a compass needle—it will set itself in the direction of the horizontal component of the earth's field. The vertical plane through the axis of the compass needle is the **magnetic meridian.**

If a magnetised steel rod be freely suspended *about its centre of gravity* it will set in the magnetic meridian, but the N pole of the magnet will dip towards the earth when in the northern hemisphere, and the S pole will dip when in the southern hemisphere.

The direction of the rod will give the direction of the lines of force of the earth's magnetic field at the place considered.

The earth's field in any given locality is the same over a region which is of considerable extent, as compared with the dimensions of artificial magnets. For instance, if there is no iron or magnetic matter in a room or the immediate neighbourhood, the lines of force of the earth's field may be considered as uniform throughout the region considered (see Chapter IX.).

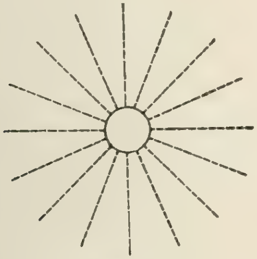

Fig. 91.

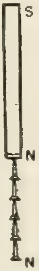

Fig. 92.

80. Induced Magnetism.—Suppose a bar magnet is supported vertically, and a number of small pieces of iron wire or small iron nails be taken. Let the end of one nail be brought to the lower end of the magnet (Fig. 92). It will be found that the nail will cling to the end of the magnet, and it is said to have become *magnetised by induction*, so that if the pole of the magnet is N, the end of the nail in contact becomes a S pole, and the other end of the nail a N pole. If, now, the end of a second nail be brought to the lower end of the first suspended nail, it is found that the second nail will hang to the first, having become magnetised by induction. In this way it is possible to have a chain of small nails from the end of the magnet.

A further illustration of magnetisation by induction is obtained by means of the iron filings method of picturing magnetic fields. Suppose in Fig. 93 two bar magnets are arranged with their unlike poles facing and a small distance apart. If, now, an originally unmagnetised bar of soft iron be placed in the space between the two poles, and a picture of the magnetic field taken by means of iron filings, the lines of force will be as shown in the lower diagram of Fig. 93.

By comparing the two diagrams of Fig. 93 it is seen that the lines of force tend to crowd into the soft iron, producing a S pole at the end facing the N pole of the bar magnet, and a N pole at the end facing the bar magnet. The soft iron has become magnetised by induction by the magnetic field between the poles of the two magnets. The fact that the lines of force tend to crowd to the soft iron from the surrounding space is explained by saying that the soft iron is a better conductor for the lines of force, or is more *permeable*, or has greater **permeability** (see also §§ 108 and 115).

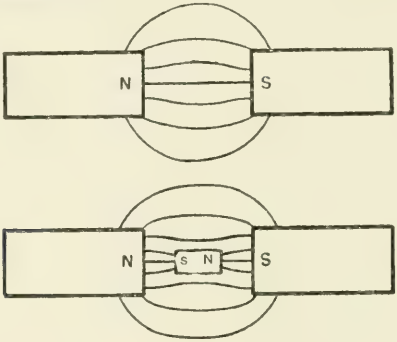

FIG. 93.

81. Precise Definition of Magnetic Poles.—When a magnet is suspended horizontally in a uniform field, the forces acting on each element of magnetism of the magnet are parallel to each other. The forces act on the positive magnetism in one direction, and on the negative magnetism in the opposite direction. All the forces which act on the positive magnetism may be replaced by a single force, equal in magnitude to the sum of the individual forces, and acting at a point which may be considered the centre of action of the positive magnetism. Similarly, all the forces acting on the negative magnetism may be replaced by a single force equal to the sum of the individual forces, and acting through a point in the magnet which may be termed the centre of action of the negative magnetism. Thus, all the forces on the magnet may be replaced by two equal opposite parallel forces applied at two fixed points in the magnet.

These points may be defined as the poles of the magnet, and the line joining these points the *magnetic axis*.

This definition of the poles is only applicable for the case in which a magnet is placed in a uniform field, such as the earth's field.

Generally, however, when considering the action of one magnet on another, it is not permissible to treat the magnetism as though it were concentrated at definite points.

82. Electro-Magnets.—If a coil of insulated wire be wound round a bar (Fig. 94), and a direct current of electricity passed through the coil, it is found that the bar becomes powerfully magnetised. If the current circuit is then broken, the bar is found to retain its magnetism to a very considerable extent.

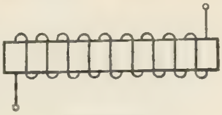

FIG. 94.

The mutual action between electric currents and magnets will be considered in detail later on (see Section IV.), and it is sufficient to state here that the usual method of producing magnets is by means of an electric "solenoid"—that is, a coil of insulated wire wound on former (*e.g.*, a cylindrical former), and excited by means of direct current, the iron or steel to be magnetised being enveloped by the solenoid.

83. Magnetic Materials other than Iron and Steel.—Whilst magnets made of soft iron or steel are enormously more powerful than magnets of any other material, yet there are two substances—viz., **cobalt** and **nickel**—which are noticeably magnetic. Pieces of these metals fly to and are held by strongly magnetised iron and, in fact, it has been found possible to make a compass needle of hardened nickel. (For more detailed information of the magnetic properties of cobalt and nickel, see Chapter VIII., § 120, Fig. 150.)

CHAPTER VII

COULOMB'S LAW OF THE FORCE BETWEEN MAGNETIC POLES—DEDUCTIONS FROM COULOMB'S LAW

84. Coulomb's Law.—It was pointed out in § 73 that magnetic poles of like polarity repel each other, and magnetic poles of unlike polarity attract each other. This mutual action between magnetic poles was investigated by Coulomb by means of the Torsion Balance (§ 7).

As has been stated in Chapter VI., it is not possible to obtain an isolated magnetic pole, but a close approximation is obtained by using a very long thin magnet in which the poles are concentrated near the ends. In the case of such a magnet the influence of one pole in the neighbourhood of the other is sufficiently small to be neglected. In this way Coulomb established the result that the *force between two magnetic poles is inversely proportional to the square of the distance between them.* Another method of proving this result is given in § 97.

In order to determine the influence of the *strength of each pole* on the force between them, the procedure is not quite so simple as was the corresponding problem for the case of electric charges, in which it was easy to subdivide the charge, and so determine the effect on the force of increasing or decreasing the charge.

In the case of magnets, however, a difficulty arises. For suppose there are two precisely similar magnetic poles A and B, and the force of each in turn on a third pole C distant d be measured. When the poles A and B are placed together at the same distance from C the resultant force on C is not double that found for A and B acting separately, but is less than double owing to the weakening action of A and B on each other.

It is possible, however, to determine indirectly the effect of the strength of two poles on the mutual force between them as follows:

Suppose there is a series of long thin magnets, the poles of which are concentrated near the ends, and the pole strengths, or amount of magnetism, of which are respectively $m_1 : m_2 : m_3$. If by means of the Torsion Balance the force is found between m_1 and $m_2 : m_3 : \ldots$ in turn, then betweeen m_2 and $m_1 : m_3 : \ldots$ in turn, then between m_3 and $m_1 : m_2 : \ldots$ in turn, the distance apart of the poles between which the force is measured being in all cases the same, it is found that the force is given by the product of two quantities, of which one is characteristic of one pole and the other is characteristic of the other pole. These quantities measure the *pole strengths* of the respective poles (see § 85).

The above results may be generalised as follows:

> **The force between two magnetic poles is directly proportional to the strength of each pole, and inversely proportional to the square of the distance between them.**

Expressed in symbols this may be written—

$$F=\frac{m_1 m_2}{\mu d^2};$$

where m_1 and m_2 are the respective pole strengths, and d is the distance between them. The factor μ is a constant, and, as will be seen later (§ 111), represents the **permeability** of the medium through which the force acts.

The question of the nature and absolute value of μ involves the question of the nature of magnetism. For the present it is only necessary to deal with relative values of the permeability, and for comparative purposes the value of μ for air or a vacuum is taken to be unity.

85. Unit Pole.—Suppose two precisely similar poles are placed at a distance 1 cm. apart in air, and suppose the poles are such that the force between them is 1 dyne. Then in the formula—

$$F=\frac{m_1 m_2}{\mu d^2},$$

$$F=1 \text{ dyne} : \mu=1 : d=1 \text{ cm.} : m_1=m_2.$$

Hence $$m_1^2=m_2^2=1:$$

and $$m_1=m_2=1;$$

that is to say, each pole is of unit strength, the system of units being such that μ is taken to be unity, the centimetre is the unit of length, the gramme the unit of mass, and the second the unit of time. Hence—

> **Unit magnetic pole is such that, when placed in air at a distance of 1 cm. from a precisely similar pole, it is repelled with a force of 1 dyne.**

Usually a unit N pole is understood when reference is made to a unit pole.

As regards the definition of unit pole, it is to be observed that in practice it would seldom be permissible to bring the two poles as close as a centimetre apart, owing to the mutual weakening action which would take place. Moreover, the question would arise as to between precisely which points in the poles the distance should be measured.

The difficulty, however, is only apparent, for the poles could be taken at, say, a distance of 10 cms. apart, at which distance the force between unit poles would be $\frac{1}{100}$ dyne.

The law is to be taken as expressing the result that the force between two elements of magnetism considered as being concentrated at a point is directly proportional to the product of the amount of magnetism of each element, and inversely proportional to the square of the distance between them.

86. Magnetic Force or Intensity of Magnetic Field.—The magnetic force, or, as it is often called, the intensity of magnetic field, at a point in a magnetic field is the force in dynes which a unit pole would experience if placed at that point, and the direction of the force is the direction in which a unit N pole would tend to move when placed at that point. It is to be assumed that the introduction of the unit pole does not disturb the distribution of magnetism to which the field is due.

If the field is relatively weak, the force may be defined as, say, 100 times the force experienced by a pole of strength $\frac{1}{100}$ of a unit pole when placed at that point.

The magnetic force at a point in a field is usually denoted by the letter H.

The force due to a pole of strength m at a point distant r cms. in air is

$$\frac{m}{r^2} \text{ dynes.}$$

For purposes of calculation the force at any point in a non-magnetisable substance may be calculated from the expression

$$\Sigma \frac{\delta m}{r^2},$$

where δm is an element of magnetism distant r cms. from the point in question, and the sign Σ denotes the sum of the forces due to all such elements.

More generally stated, the force at a point in a magnetic field is

$$\Sigma \frac{\delta m}{r^2},$$

where the summation refers to all the magnetism present throughout the field — that is, including the magnetism developed in any part of the medium which is magnetic.

As will be seen later, this is equivalent to saying that the force due to a pole m and immersed in a magnetic medium is

$$\frac{m}{\mu r^2} \text{ dynes,}$$

where μ is the permeability of the medium (§ 111).

87. Unit Line of Force.—Consider a uniform magnetic field of unit intensity as defined in § 86. If a plane surface be drawn in the field and perpendicular to the direction of the lines of force (see § 77), then one unit line of force is said to cross each sq. cm. of area of the surface.

The magnetic intensity at any point in a field may therefore be defined as the number of unit lines of force per sq. cm. which pass through an elementary plane surface containing the point and drawn perpendicular to the lines of force.

If at any point in a magnetic field X is the density of the lines of force as defined above, the magnetic intensity is sometimes defined as X *gauss*, although the term "gauss" is seldom used in practice.

88. Tube of Force.—If a small closed curve is drawn at any place in a magnetic field, and if lines of force are drawn through the curve at every point of the curve, the lines so drawn will enclose a tubular space termed "a tube of force."

89. Flux of Force.—If an element of surface δS be drawn in a magnetic field at right angles to the direction of the magnetic force at that place, and if H is the intensity of the force at every point of the elementary surface δS the product

$$H\delta S$$

is termed "the flux of force through the area δS," the element δS being taken sufficiently small so that the force H may be considered constant at every point of the element of surface.

Since the law of magnetic action between magnetic poles is precisely the same as the law of electric action between electric charges, Gauss' theorem (§ 20) holds for magnetic flux—viz., that the magnetic flux through a closed surface is zero, and this is true, even if the surface encloses magnets, since the total magnetism of a magnet is always zero (§ 76).

Consider a tube of force bounded by the surfaces δS_1 and δS_2, which are respectively at right angles to the directions of the intensities H_1 and H_2 (Fig. 95). Then, by definition of a tube of force, the magnetic flux is constant throughout the tube, or

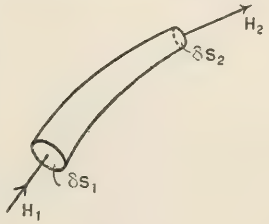

FIG. 95.

$$H_1\delta S_1 = H_2\delta S_2;$$

that is,
$$\frac{H_1}{H_2}=\frac{\delta S_2}{\delta S_1},$$

or the magnetic intensity is inversely proportional to the cross-section of the tube.

Suppose a unit magnet pole is at the centre of a sphere of 1 cm. radius. The magnetic intensity H is unity at every point on the surface of the sphere. Hence the total flux through the surface of the sphere is equal to the area of the surface of the sphere, or 4π unit lines. *Hence unit magnetic pole gives rise to* 4π *unit lines of force.*

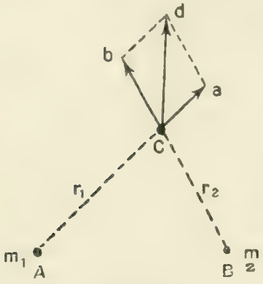

Fig. 96.

90. Superposition of Magnetic Fields.—Suppose there are two magnetic masses (*i.e.*, quantities of magnetism), $+m_1$ and $+m_2$, situated respectively at A and B (Fig. 96), and suppose it is desired to find the magnitude and direction of the resultant force at a point C distant r_1 and r_2 respectively from A and B.

The force at C due to m_1 at A is in the direction AC, and is equal to

$$\frac{m_1}{r_1^2} \text{ dynes.}$$

Let it be represented by the vector C*a*.

The force at C due to m_2 at B is in the direction BC, and is equal to

$$\frac{m_2}{r_2^2} \text{ dynes.}$$

Let it be represented by the vector C*b*.

The resultant force at C is then given in magnitude and direction by the diagonal *cd* of the parallelogram *cbda*. That is to say, the force at C due to each magnetic mass at A and B acts as though the other were absent, the resultant being obtained by superposing the two forces. This is experimentally verifiable, and is true whatever distribution of magnetic masses is assumed, and whatever the medium through which they act. It is to be observed that if the forces act through a magnetisable medium, the effect of the induced free magnetism must be included in determining the resultant force C*a* at C due to m_1, and the resultant force C*b* at C due to m_2.

Similarly for fluxes of magnetic force. If through any closed curve the flux in any direction due to mass m_1 at A is F_1 when m_1 is the only mass present, and if through the same closed curve the flux in the same direction is F_2 due to a mass m_2 at B when m_2 is the only mass present, then, other conditions such as the permeability of the medium remaining the same, the flux in the given direction through the same closed curve when mass m_1 is at A and m_2 is at B is F_1+F_2.

This principle of superposition is of great importance, an example of its application being the determination of the resultant magnetic field in the air gap of a dynamo.

91. Magnetic Moment of a Magnet.—Suppose a magnet of pole strength m and distance between the poles l cms. (as defined in § 81) is free to move in a uniform magnetic field of intensity H, for example, in the earth's field. The magnet will set itself with its axis in the direction of the field.

If the magnet is placed with its axis at right angles to the direction of the field, the forces on it will form a couple of moment Hml (see Fig. 97).

If the intensity of the uniform field is unity—*i.e.*, $H=1$—the couple on the magnet when placed at right angles to the field will be ml. This quantity is called the **magnetic moment** of the magnet, and is usually denoted by the letter M.

The moment of a magnet is, therefore, the moment of the couple which would act on the magnet if placed in a uniform field of unit intensity, the axis of the magnet being at right angles to the direction of the field. The moment of the magnet is equal to the pole strength multiplied by the distance between the poles.

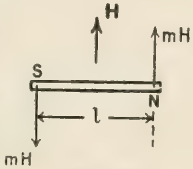

FIG. 97.

92. Uniform Magnetisation.—A magnet is said to be uniformly magnetised if an element of volume taken at any part has the same magnetic moment and has the same direction of magnetic axis.

93. Intensity of Magnetisation of a Magnet.—Consider a magnet uniformly magnetised as defined in § 92. If S sq. cms. is the area of the pole and m the pole strength, then the amount of magnetism per sq. cm.—*i.e.*, the density of magnetism—is

$$\frac{m}{S}=I,$$

where I is the **intensity of magnetisation.**

The above expression may be written—

$$I=\frac{ml}{Sl},$$

where l cms. is the length magnet.

But ml is the magnetic moment M, and Sl cubic cms. is the volume of the magnet.

Hence
$$I=\frac{M}{\text{volume}},$$

or I = magnetic moment per cubic cm.

If the magnet is not uniformly magnetised, the intensity of magnetisation at any point in the magnet is—

$$I=\frac{\delta M}{\delta v},$$

where δM is the magnetic moment of the element of volume δv in which the point lies.

94. Resultant Moment of a Number of Magnets rigidly Fixed Together.—Suppose a number of magnets of pole strengths $m_1: m_2: m_3: \ldots$ respectively rigidly fixed together and placed in a uniform field of intensity H (Fig. 98). The action of the field on, say, the magnet m_1 will be a couple of moment $m_1Hl_1 \cos\theta_1$, and similarly for the other magnets.

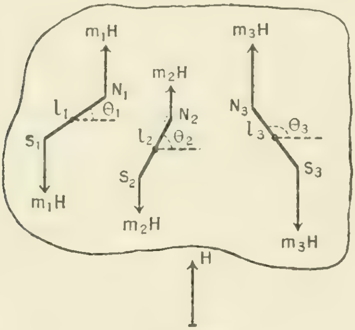

Fig. 98.

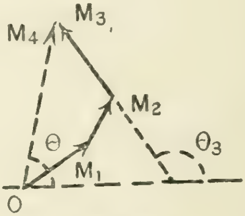

Fig. 99.

The resultant of all these couples will be a single couple, whose moment is equal to the algebraical sum of the individual couples—viz.:

$$\begin{aligned}&H(m_1l_1\cos\theta_1+m_2l_2\cos\theta_2+m_3l_3\cos\theta_3)\\=&H(M_1\cos\theta_1+M_2\cos\theta_2+M_3\cos\theta_3),\end{aligned}$$

where $M_1: M_2: M_3$ represent the moments of the individual magnets.

Now suppose lines are drawn parallel to the axis of each magnet, and of length proportional to the moment of that magnet, and so that they form a polygon (Fig. 99). The closing line of the polygon will give the resultant or equivalent moment of the system, and the system will set so that this resultant is in the direction of the field H.

That this is so may be seen by observing that the projection of the resultant M_4 on any line is equal to the sum of the projections of the individual sides of the polygon—

that is $M_1 \cos \theta_1 + M_2 \cos \theta_2 + M_3 \cos \theta_3 = M_4 \cos \theta$.

The closing line OM_4 gives the magnitude of the moment and direction of axis of the equivalent magnet.

If the relative direction of the individual magnets and the respective moments are such that the polygon closes on itself, the system will be in neutral equilibrium—that is to say, it will stay in any position, and such a system is said to be **astatic.**

A system consisting of two needles suspended parallel to one another, and in the same plane with their unlike poles adjacent, is used for astatic galvanometers in which it is desired to eliminate the action of the earth's field (see § 208).

95. The Potential Energy of a Magnet brought into a Uniform Magnetic Field.— The potential energy of a magnet in a magnetic field is the work which the magnet can perform in moving out of the field.

Suppose a magnet of moment $M = ml$ is brought from an indefinitely great distance into a uniform field of strength H, the axis of the magnet being kept at right angles to the direction of the field during the transfer. The total work done in making the transfer will be zero, since the work *against* the action of the field in moving up one pole is equal to the work done *by* the action of the field in moving up the other pole (see Fig. 100).

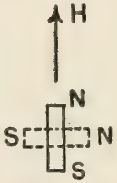

Fig. 100.

Now let the magnet be turned so that its axis lies in the direction of the field. Since the length of the magnet is l, the work done during this movement *by* the field on the N and S poles of the magnet respectively will be

$$mH \frac{l}{2} \text{ ergs},$$

or the total work done *by* the field will be mlH—that is,

$$MH \text{ ergs}.$$

Hence the potential energy of the magnet in the field is—

$$-MH \text{ ergs}.$$

If the magnet be turned through 180° so that its axis is directly opposed to the direction of the field, the potential energy of the magnet in the field will be—

$$MH \text{ ergs}.$$

These results can be expressed more generally as follows: The potential energy of a magnet of moment M introduced into a uniform field of intensity H is

$$-MH \cos \theta \text{ ergs,}$$

where θ is the angle which the direction of the field makes with the axis of the magnet (see Fig. 101). The potential energy of the magnet in the field is thus a minimum when θ is equal to zero.

From a consideration of the force on the poles of the magnet (Fig. 101), it appears that if the magnet is free it will move so that θ becomes zero—that is, so that the potential energy of the magnet in the field becomes a *minimum*.

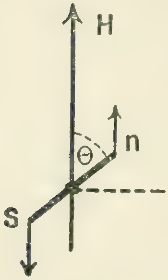

Fig. 101.

96. The Force on a Small Magnet in a Non-uniform Magnetic Field.—Consideration of the forces on the respective poles of the small magnet shows that the magnet will set with its axis in the direction of the field.

Let F dynes be the resultant of the forces on the poles of the small magnet. The direction of this force will be such that the magnet will tend to move in the direction corresponding to the maximum rate of diminution of energy of the magnet, just as a body falling under gravity moves in the direction of maximum decrease of energy.

Let δH be the *increase* of the intensity of the magnetic field into which the small magnet moves when slightly displaced by an amount δr cms. in the direction of F.

Then $M\delta H$ is the corresponding diminution of energy of the small magnet in the field (see § 95) and therefore—

$$M\delta H = F\delta r.$$

The magnet will therefore move from the weak to the strong part of the field, the direction of movement being that direction corresponding to maximum rate of decrease of energy of the magnet.

Thus, in Fig. 89, let a small magnet be placed in the outer part of the field with its centre equidistant from each pole of Fig. 89, and with its axis parallel to the direction of the lines of force of the field at that place.

Equal and opposite forces will act on the poles of the small magnet and the resultant force will act through the centre of the magnet and in a direction perpendicular to the lines of force of the original field. The small magnet will therefore tend to move in a direction *perpendicular* to the direction of the field in which it is placed, from a weak to a stronger part of the field.

If a small piece of soft iron is placed in the non-uniform magnetic field the soft iron becomes magnetised by induction and the result will be similar to that just considered for a small magnet.

In the case of the soft iron magnetised by induction the energy of the soft iron in the magnetic field will be $-\frac{1}{2}$MH, if it be assumed that the magnetic moment M of the soft iron is directly proportional to the intensity H.

If a small magnet is held with its axis making 180° with the axis of the field (*i.e.*, directly opposed to the field), it tends to move from the strong to the weak part of the field.

Further considerations as to the energy of small magnets in a magnetic field will be found in § 99.

97. Gauss' Proof that the Force between Two Magnetic Poles is inversely as the Square of the Distance between Them.—Let *ns* be a small bar magnet (Fig. 102), of which the magnetism may be considered as concentrated at the ends *ns*. Such a condition may be practically approximated if the lateral dimensions of the magnet are small compared with the length. Consider the intensity of the force due to the magnet at two points, P and Q, respectively on a line through the axis of *ns*, and on a line perpendicular to the axis of *ns* and passing through the middle point.

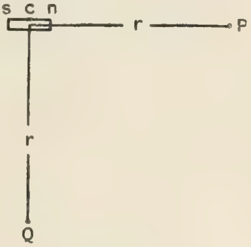

Fig. 102.

Let m be the strength of each pole of the magnet *ns*, and let l cms. be the distance between the poles.

Let CP$=$CQ$=r$ cms., and let the distance r be large compared with l, the length of the magnet.

Case I.—Consider the magnetic intensity at a point P lying on the prolongation of the axis of the magnet. The force due to the north pole of *ns* is (§ 86)—

$$\frac{m}{\left(r-\frac{l}{2}\right)^2} \text{ dynes,}$$

and the force due to the south pole of *ns* is—

$$\frac{m}{\left(r+\frac{l}{2}\right)^2} \text{ dynes.}$$

The resultant force at P is—

$$F=m\left[\frac{1}{\left(r-\frac{l}{2}\right)^2}-\frac{1}{\left(r+\frac{l}{2}\right)^2}\right] \text{ dynes}$$

$$=\frac{m2lr}{\left(r^2-\frac{l^2}{4}\right)^2}.$$

Since l is small compared with r, the resultant force at P is, very approximately,

$$F=\frac{2ml}{r^3}$$

$$=\frac{2M}{r^3}\text{ dynes,}$$

where M is the magnetic moment of the magnet.

Case II.—Consider the magnetic intensity at a point Q lying on a line through the centre C of the magnet and perpendicular to the axis of the magnet (Figs. 102 and 103).

The force due to the north pole of *ns* is—

$$F_n=\frac{m}{r^2+\frac{l^2}{4}}\text{ dynes.}$$

The force due to the south pole of *ns* is—

$$F_s=\frac{m}{r^2+\frac{l^2}{4}}\text{ dynes.}$$

The resultant force at Q is (see Fig. 103)—

$$F=F_n\cos\alpha+F_s\cos\alpha$$

$$=\frac{2m}{r^2+\frac{l^2}{4}}\cos\alpha$$

$$=\left[\frac{2m}{\left(r^2+\frac{l^2}{4}\right)}\right]\left[\frac{l}{2\left(r^2+\frac{l^2}{4}\right)^{\frac{1}{2}}}\right]$$

$$=\frac{ml}{\left(r^2+\frac{l^2}{4}\right)^{3/2}}$$

$$=\frac{ml}{r^3\left(1+\frac{l^2}{4r^2}\right)^{3/2}}$$

$$=\frac{ml}{r^3}\left(1-\frac{3}{2}\times\frac{l^2}{4r^2}+\ .\ .\ .\right).$$

Since l is small compared with r, the term $\frac{3}{8}\frac{l^2}{r^2}$ and succeeding terms may be neglected. Hence the force at Q is—

$$F=\frac{ml}{r^3}$$

$$=\frac{M}{r^3}\text{ dynes,}$$

where M is the magnetic moment of the magnet.

The intensity of the force at Q is, therefore, half that at P.

It can be shown, generally, that if the force between two magnetic poles were to vary inversely as the nth power of the distance between them, the intensity of the force at Q would be $\frac{1}{n}$ that at P.

In order to test whether the force at P is double that at Q, the following experiment may be made.

In Fig. 104 let n_1s_1 be a small compass needle of magnetic moment X, and let AB be the direction of the magnetic meridian. If no magnetic material is in the neighbourhood of the compass needle, its axis will lie along AB.

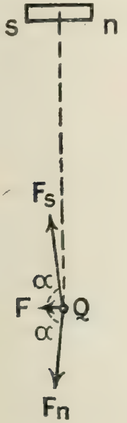

FIG. 103.

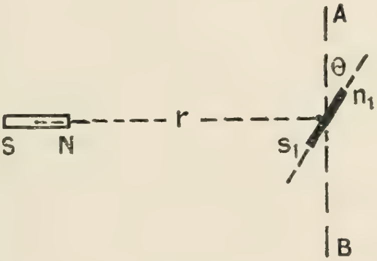

FIG. 104.

Let a small bar magnet NS of length l cms. be placed so that its axis lies on a line through the centre of the compass needle and perpendicular to AB. The bar magnet is then said to be in the "end on" position.

Let the distance r cms. between the centres of bar magnet and compass needle be great compared with the length of either compass needle or bar magnet.

The compass needle will be deflected so that it makes an angle θ with the meridian. The forces on the needle are:

(i.) A couple XH sin θ tending to restore it to the meridian, where H is the horizontal component of the intensity of the earth's field (see § 94).

(ii.) A couple

$$\frac{2mlX\cos\theta}{r^3}$$

tending to set it at right angles to the meridian (see Case I., p. 115).

These couples must be equal.

Hence
$$\frac{2ml\mathrm{X}\cos\theta}{r^3}=\mathrm{XH}\sin\theta;$$

or
$$\tan\theta=\frac{2ml}{\mathrm{H}r^3}=\frac{2\mathrm{M}}{\mathrm{H}r^3},$$

where M is the magnetic moment of the bar magnet.

Next let the magnet NS be placed either north or south of the compass needle, so that the centre of the bar magnet lies in the magnetic meridian AB, and its axis at right angles to the meridian (Fig. 105). The magnet is then said to be in the "broadside on" position. Let θ be the deflection of the compass needle in this case.

The needle will now be in equilibrium under the action of two couples—viz.:

(i.) $\mathrm{XH}\sin\theta$ due to the earth's field.

(ii.) $\frac{\mathrm{XM}\cos\theta}{r^3}$ due to the action of the magnet NS.

These couples must be equal.

Therefore
$$\tan\theta=\frac{\mathrm{M}}{\mathrm{H}r^3}.$$

Hence $\tan\theta$ for Fig. 104 is equal to $2\tan\theta$ for Fig. 105.

If the deflections θ of the compass needle are small this may be written:

θ for "end on" position (Fig. 104) $=2\theta$ for "broadside on" position (Fig. 105).

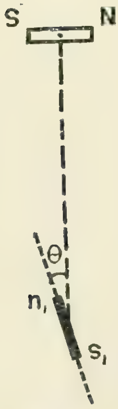

FIG. 105.

Experiment shows that this relationship holds, and hence gives a verification of Coulomb's Law that the force between two magnetic poles varies inversely as the square of the distance between them.

98. Magnetic Potential at a Point in a Magnetic Field.—The law of force between magnetic masses has been shown to be precisely the same as that between electric charges (see §§ 7 and 84). If unit magnetic mass be brought up from an infinite distance to a point in a magnetic field, either work will have to be supplied in overcoming the opposing magnetic forces during the transfer of the unit magnetic mass, or work will be done by the magnetic forces during the transfer.

In any case the work done in bringing up a unit positive magnetic mass from infinite distance to a point in a magnetic field is the **potential** at that point of the field. If the work is done *against* the magnetic forces, the potential at the point is *positive*; if work is done *by* the magnetic forces, the potential at the point is *negative.*

It may be proved, by a similar method to that used for the case of the electric field, that the work done in moving the unit charge between any two points in a magnetic field does not depend on the path taken in moving between the two points, but only on the magnetic conditions of the two points (see § 29).

The relationship between the force F in any direction x and the potential V at a point in a magnetic field can be shown to be—

$$F = -\frac{dV}{dx}$$

in a precisely similar manner as in the corresponding case of an electric field (§ 30).

The magnetic force in any direction is, therefore, equal to the rate of decrease of the potential in that direction.

Consequently, the magnetic force at any point in a magnetic field acts in the direction corresponding to maximum rate of decrease of potential.

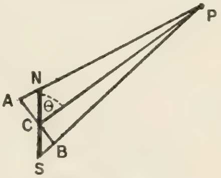

FIG. 106.

99. Potential at a Point P in the Magnetic Field due to a Small Magnet, the Point P being at a Distance from the Centre of the Magnet which is Large compared with the Length of the Magnet.—In Fig. 106 let NS be the magnet, and assume that the magnetism is concentrated at the two ends N and S. Let m be the pole strength and l cms. the length of NS, and let r cms. be the distance of P from the centre C.

The potential at P due to the N pole is (see §§ 32 and 84)—

$$\frac{m}{NP},$$

the permeability μ being unity.

The potential at P due to the S pole is (see §§ 32 and 84)—

$$-\frac{m}{SP}.$$

The resultant potential at P due to both poles is—

$$V = \frac{m}{NP} - \frac{m}{SP}.$$

Draw ACB perpendicular to CP, and draw PNA and PBS. Now if CP is very large compared with NS the lines NA and SB will be very approximately perpendicular to AB; also PA, PB, and PC will be very approximately equal—that is, PA = PB = r very approximately.

Hence $$\mathrm{PN}=\mathrm{PA}-\mathrm{AN}=r-\frac{l}{2}\cos\theta,$$

$$\mathrm{PS}=\mathrm{PB}+\mathrm{BS}=r+\frac{l}{2}\cos\theta.$$

Therefore $$\mathrm{V}=\frac{m}{r-\frac{l}{2}\cos\theta}-\frac{m}{r+\frac{l}{2}\cos\theta};$$

or $$\mathrm{V}=\frac{ml\cos\theta}{r^2-\frac{l^2}{4}\cos^2\theta}$$

$$=\frac{\mathrm{M}\cos\theta}{r^2},$$

where M is the moment of the magnet NS and $\frac{l^2}{4}$ is assumed to be negligibly small as compared with r^2.

Since the magnetic force is given by

$$\mathrm{F}=-\frac{d\mathrm{V}}{dr},$$

the formulæ just deduced for V may be used to obtain the results given in § 97—viz.: (1) the force at a point in the axis of a magnet at a relatively great distance r from the centre of the magnet; and (2) the force at a point on a line through the centre of the magnet and perpendicular to the axis. Thus:

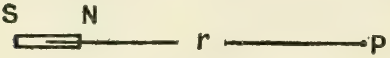

Fig. 107.

(1) *The force at P* (Fig. 107) *on the axis of the magnet.*

For such a point the angle θ of Fig. 106 is zero, and the potential is therefore

$$\frac{\mathrm{M}}{r^2}.$$

Hence $$\mathrm{F}=-\frac{d\mathrm{V}}{dr}$$

$$=\frac{2\mathrm{M}}{r^3}\text{ dynes.}$$

(2) *The force at P* (Fig. 108) *on a line through the centre of the magnet and perpendicular to the axis.*

For *any* point P on such a line the potential is zero, since the value of θ of Fig. 106 is 90°, and consequently $\cos\theta=0$.

It follows, therefore, that there is no force at P in the direction PC, and the force at P due to the magnet NS must therefore be in a direction at right angles to PC.

Let δx be a small distance measured from P in a direction at right angles to PC.

Then $$\delta x = r\delta\theta,$$

and the force in the direction at right angles to PC is—

$$F = -\frac{dV}{dx} = -\frac{dV}{rd\theta}.$$

But, as shown above, $$V = \frac{M\cos\theta}{r^2}.$$

Therefore $$F = \frac{M\sin\theta}{r^3}$$
$$= \frac{M}{r^3},$$

since $\theta = 90°$.

Now suppose the magnet is a small cylindrical bar magnet, in which the magnetism is concentrated on the end faces (Fig. 109). Let δS be the area of each end, I the density of the magnetism (see § 93), and l cms. the length of the bar. The potential at a point P in the field of this magnet is—

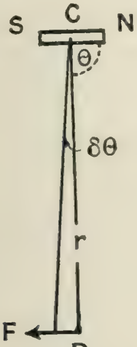

FIG. 108.

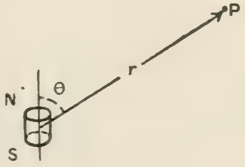

FIG. 109.

$$\frac{M\cos\theta}{r^2} = \frac{lI\delta S\cos\theta}{r^2},$$

where θ is the angle which the line joining P to the centre of the magnet makes with the axis of the magnet.

Let $\delta\omega$ be the solid angle subtended at P by either end face δS of the magnet. Since l is relatively very small, the solid angle subtended at P by each end of the cylinder is very approximately the same.

Then
$$\frac{\delta S \cos\theta}{r^2}=\delta\omega,$$

or the potential at P due to the small cylindrical bar magnet is

$$Il\delta\omega.$$

This result is of importance in connection with the consideration of the magnetic shell (§§ 103-106).

100. Magnetic Lines of Force and Equipotential Lines.—Since the force between two magnetic masses varies according to a precisely similar law as the force between two electric charges, the lines of force (and, consequently, the tubes of force) due to two magnetic masses may be constructed by exactly the same methods as explained in connection with the corresponding electrical cases given in Section I., § 19.

Thus, if the poles of a magnet are imagined concentrated at the ends of a magnet, the lines of force may be drawn as shown in § 19 (Example 2, Fig. 14).

A *magnetic equipotential surface* is one drawn so that the potential at every point of the surface is constant. By reasoning in precisely the same way as in § 33 it follows that the lines of force cut the equipotential surfaces at right angles.

101. Magnetic Shell of Uniform Strength.—Imagine a thin sheet of magnetisable substance of uniform thickness, and uniformly magnetised in a direction perpendicular to the surface of the sheet. Such a magnetised sheet of material is called a *magnetic shell.*

> **The importance of the investigation of the properties of a magnetic shell lies in the fact that such a shell is equivalent in magnetic effects to a current of a certain strength flowing in an electric circuit of which the shape is the same as the boundary edge of the shell.**

This equivalence was first stated and proved by Ampère, and all subsequent experiments confirm the statement. The results deduced for a magnetic shell—*e.g.*, potential at a point outside the shell, force on shell when placed in a magnetic field, etc.—can be immediately applied to the case of the equivalent electric circuit (see § 181).

The properties of the magnetic shell can be deduced from Coulomb's Law, and although such a shell actually does not appear in practice, the fact that it is equivalent to an electric current means that the investigation of the properties of a magnetic shell is also an investigation of the magnetic properties of an electric current, and thus allows the linking up of the phenomena of magnetism and electric currents respectively.

102. Strength of a Magnetic Shell.—The strength of a magnetic shell is the product of the intensity of magnetisation (§ 90) into the thickness of the shell or otherwise stated, is equal to the magnetic moment per unit area. If the thickness is t cms. and the intensity of magnetisation of the shell is I—this being also the density of magnetism on the surface of the shell (see § 93)—the strength of the shell is

$$\Phi = It.$$

103. Potential of a Magnetic Shell at a Point P.—To find the potential of a magnetic shell at an external point P (see Fig. 110). Consider a filament AB of the shell taken in the direction of magnetisation—*i.e.*, the perpendicular to the surface of the shell.

Let δS sq. cms. be the element of surface on each side of the shell forming the ends of the filament.

Let the point P be distant r cms. from C, the centre of the filament.

Let A be magnetised with positive magnetism of intensity $+I$ and B magnetised with negative magnetism of intensity $-I$.

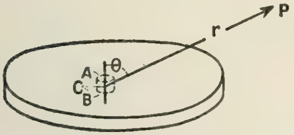

Fig. 110.

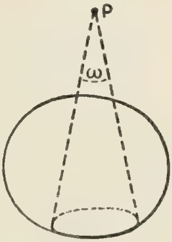

Fig. 111.

The magnetism on the end surface A of the filament is $I\delta S$, and the magnetism on the end surface B is $-I\delta S$.

Consider three cases—viz.: (1) when the point P is on the positively magnetised side of the shell; (2) when the point P is on the negatively magnetised side of the shell; and (3) when the point P is inside the material of the shell.

Case I.—The point P is on the positively magnetised side of the shell (Fig. 111).

The potential at P due to the elementary magnet, such as AB (Fig. 110), is by § 99—

$$= \frac{It\delta S \cos\theta}{r^2}$$

$$= \Phi \frac{\delta S \cos\theta}{r^2},$$

where θ is the angle CP makes with the axis CA of the magnet AB (Fig. 110), the end A being positively magnetised, as already stated.

If $\delta\omega$ is the solid angle which the end surface A of the filament AB subtends at P—

$$\delta\omega = \frac{\delta S \cos \theta}{r^2}.$$

Hence the potential at P is—

$$\Phi\delta\omega$$

i.e., the strength of the shell multiplied by the solid angle subtended at P by the element of the shell. A similar result holds for each element of the shell, and hence the potential at P due to the whole shell is equal to the strength of the shell multiplied by the solid angle subtended at P by the whole shell—*i.e.*, $\Phi\omega$.

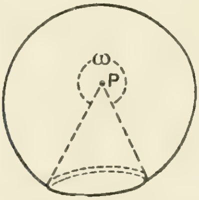

Fig. 112.

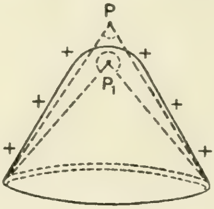

Fig. 113.

But the solid angle subtended at P by the shell is the solid angle subtended at P by the boundary of the shell.

Hence the potential at P due to the shell depends only on the strength of the shell, and the shape and position of the boundary of the shell.

It is a matter of indifference what shape the shell follows between the boundaries.

If the shell is completely closed, ω is equal to zero, and hence the potential at a point outside a completely closed shell is zero.

Case II.—The point P is on the negatively magnetised side of the shell (Fig. 112).

The potential at P due to the elementary magnet, such as AB in Fig. 110, is by § 99—

$$= -\frac{It\delta S \cos \theta}{r^2}$$

$$= -\Phi \frac{\delta S \cos \theta}{r^2},$$

where θ is now the angle CP makes with the axis CB of the magnet AB (Fig. 110), the end B being negatively magnetised and the point P on the negative side of the shell. All the other symbols have the same meaning as in Case I. The potential due to the whole shell is therefore—

$$-\Phi\omega,$$

where ω is the solid angle subtended at P by the boundary of the shell.

If the shell is completely closed, and the point P lies inside the shell, the solid angle ω will be equal to 4π, and hence the potential at a point inside a closed shell is $-4\pi\Phi$ if the inside surface of the shell is negatively magnetised. and is $+4\pi\Phi$ if the inside surface of the shell is positively magnetised.

It is to be noted that if, in Fig. 113, P and P_1 be points close together, and if P is on the positively magnetised side of the shell, and P_1 on the negatively magnetised side, then—

$$\omega=4\pi-\omega_1,$$

where ω is the solid angle subtended at P by the boundary of the shell, and ω_1 is the solid angle subtended at P_1 by the boundary of the shell.

Case III.—The point P lies inside the material of the shell (Fig. 114).

Let P be distant x from the positively magnetised side of the shell, and distant y from the negatively magnetised side of the shell.

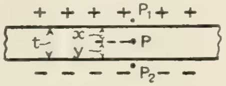

FIG. 114.

Then P may be considered as being on the negatively magnetised side of a shell of thickness x and on the positively magnetised side of a shell of thickness y, since if the shell be laminated through P the two new surfaces so produced will be negatively and positively magnetised respectively.

Hence the potential at P is—

$$\begin{aligned}&\mathrm{I}y\omega-(4\pi-\omega)\mathrm{I}x\\ &=\mathrm{I}\omega t-4\pi\mathrm{I}x\\ &=\Phi\omega-4\pi\mathrm{I}x,\end{aligned}$$

where ω is the angle subtended at a point P_1 outside the shell, and close to the positively magnetised surface, and Φ is the strength of the shell.

If $x=o$: P becomes P_1, and the potential of P is $\Phi\omega$ as given in Case I.

If $x=t$: P becomes P_2, and the potential of P is $-(4\pi-\omega)\Phi$ as given in Case II.

In passing from P_1 to P_2 the potential changes by an amount—

$$\begin{aligned}&=\omega\Phi+4\pi\Phi-\omega\Phi\\ &=4\pi\Phi.\end{aligned}$$

104. The Potential Energy of a Magnetic Pole of Strength m placed at a Point P in the Neighbourhood of a Magnetic Shell.—The potential at P is the work done by bringing up unit positive pole from an infinite distance to the point P.

As has been shown in § 103, if P is on the positively magnetised side of the shell, the potential at P is $\Phi\omega$ where Φ is the strength of the shell, and ω is the solid angle subtended by the boundary of the shell at the point P.

The work done by bringing a pole of strength m from an infinite distance to the point P is therefore (§ 98)—

$$\Phi\omega m \text{ ergs.}$$

But ωm is the magnetic flux due to the pole of strength m at P which passes through the boundary of the shell (see § 89).

Now it is clear that the potential of the pole m at P in the presence of the shell is the same as the potential of the shell in the presence of the pole m at P, since the work done in bringing up the shell from an infinite distance to the neighbourhood of the pole P is the same as bringing up the pole P from an infinite distance to the neighbourhood of the shell.

Hence, the energy of a shell placed in a magnetic field is given by the product of the strength of the shell into the magnetic flux due to the field (see § 90) which threads the boundary of the shell. The energy is positive, if the lines of force due to the field are in the direction from the positive to the negative side of the shell, and is negative if the lines of force due to the field are in the direction from the negative to the positive side of the shell.

The energy of the shell in the field will be a *minimum* if the shell is so placed that its boundary includes a maximum number of lines of force of the field in the direction from the negative to the positive side of the shell.

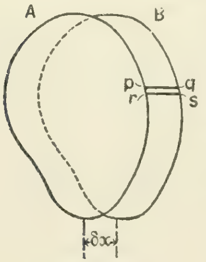

Fig. 115.

105. The Magnetic Force on a Shell placed in a Magnetic Field.—From what has been said in §§ 95 and 96, it will be seen that the shell will tend to move so that its energy in the field will become a minimum.

Let X dynes be the force on the shell in any direction x, and suppose the shell is displaced a small distance δx cms. in the direction of x (Fig. 115).

The work done by the electric force during this displacement will be—

$$X\delta x \text{ ergs.}$$

The change of energy of the shell in the field will be—

$$\Phi(N_A - N_B),$$

where N_A is the magnetic flux due to the field which threads the boundary of the shell in the positive direction before the displacement—*i.e.*, in position A; and N_B is a similar quantity which holds for the position of the shell after the displacement —*i.e.*, in position B (see § 104).

Since the force acts in the direction tending to reduce the energy of the shell in the field it follows that N_B is less than N_A.

Equating the expression for the work done on the shell and the loss of energy of the shell—

$$X\delta x = \Phi(N_A - N_B).$$

The flux $N_A - N_B$ will pass through such elements as *pqrs* forming the sides of the volume swept out by the shell during the displacement from position A to position B.

Let H be the intensity of the flux which passes in the outward direction through an elementary strip, such as *pqrs*.

Let H_N be the component of H normal to the elementary strip *pqrs*. Then if δS sq. cms. be the area of the strip *pqrs*—

$$\Sigma H_N \delta S = N_A - N_B,$$

that is $$X\delta x = \Phi(N_A - N_B) = \Phi \Sigma H_N \delta S.$$

But $$\delta S = pq \times pr \sin\theta = \delta x \times pr \sin\theta;$$

where θ is the angle between *pr* and *pq*.

Hence $$X\delta x = \Phi \Sigma H_N \delta x \times pr \sin\theta.$$

Now δx is the same for all the elementary strips such as *pqrs*.

Therefore $$X = \Phi \Sigma H_N\, pr \sin\theta.$$

Also $$H_N = H \cos\phi,$$

where ϕ is the angle between the direction of H and the normal to *pqrs*.

Therefore $$X = \Phi \Sigma H pr \cos\phi \sin\theta.$$

Hence the total force X on the shell is the same as if each element *pr* of the edge of the shell were subject to a force—

$$\Phi H \cos\phi \sin\theta\ pr \text{ dynes},$$

or the force *per unit length* of the boundary of the shell in the direction x may be taken as

$$\Phi H \cos\phi \sin\theta \text{ dynes}.$$

If the direction of x is such that $\theta = 90°$, the force is a maximum in that direction. The force, therefore, acts on the element *pr* in a direction perpendicular to *pr*. Hence, if a rectangle be constructed (Fig. 116) on the element *pr* of the edge of the shell as one side, the other side *pq* will give the direction of the force which may be taken as acting at the middle of the element. This, however, does not completely define the direction in which the force acts, because such a rectangle may be turned through 360° round *pr* as a hinge, and the direction *pq* would in all such positions be at right angles to *pr*.

Reference to the expression for the force on *pr* in which $\sin\theta$ is put equal to unity—viz.:

$$\Phi \mathrm{H} \cos\phi,$$

shows that the force will be in that direction for which $\cos\phi$ is a maximum—that is, for ϕ a minimum.

The position of the rectangle for which ϕ is a minimum gives the position which completely defines the direction of the force on *pr*. A little consideration will show that the position will be that in which the rectangle is perpendicular to the direction of the field H—that is, when the rectangle is perpendicular to the plane parallel to H and containing *pr* (see Fig. 117)—for in that position ϕ is minimum.

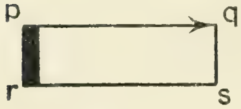

FIG. 116.

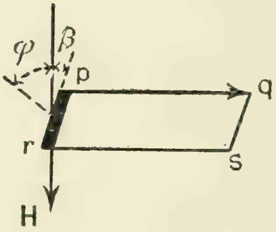

FIG. 117.

The force on the element may therefore be written—

$$\mathrm{F} = \Phi \mathrm{H} \delta s \cos\phi = \Phi \mathrm{H} \delta s \sin\beta \text{ dynes},$$

where β is the angle between *pr* and the direction of H and δs cms. is the length of the element *pr*.

The direction of the force is at right angles to the element pr and is also at right angles to the direction of H, and is such as to tend to cause as many unit lines of force as possible to pass through the shell from the negatively magnetised side to the positively magnetised side—that is, to make the energy of the shell in the field a minimum.

The importance of the result deduced here will be seen when dealing with the action of magnetic fields on current carrying circuits (see § 186).

106. Mutual Energy of Two Magnetic Shells.—Suppose the field H is due to a second shell of strength Φ_2. The potential energy of the first shell in the field of the second shell is—

$$\Phi_1 \mathrm{M}_2 \Phi_2,$$

where Φ_1 is the strength of the first shell, and $M_2\Phi_2$ is the flux due to the second shell which passes through the boundary of the first shell in the direction from the positive to the negative side of the shell (see § 104).

The potential energy of the second shell in the field of the first shell is—

$$\Phi_2 M_1 \Phi_1,$$

where $M_1\Phi_1$ is the flux due to the first shell which passes through the boundary of the second shell in the direction from the positive to the negative side of the shell.

Now it is clear that the energy of each shell in the field due to the other must have the same value, since the work done in bringing the shells into position is the same whichever shell is brought into position first.

Hence $$\Phi_1 M_2 \Phi_2 = \Phi_2 M_1 \Phi_1.$$

That is, $$M_2 = M_1.$$

M_2 is the flux due to the second shell of unit strength which threads the boundary of the first shell in the direction from the positive to the negative side of the shell, and M_1 is the flux due to the first shell of unit strength which threads the boundary of the second shell in the direction from the positive to the negative side of the shell.

It follows that if there are two neighbouring magnetic shells each of unit strength, the flux of force due to one which passes through the other in the direction from the positive to the negative side of the shell is the same for both (see § 90).

$M_2 = M_1 =$ say, M, is termed the *coefficient of mutual induction* of the two shells (see also Section IV., § 206).

The value of the coefficient of mutual induction of two magnetic shells depends therefore only on the shape of the boundary edges of the two shells and on their relative positions, but does not depend on the strength of either shell.

CHAPTER VIII

MAGNETIC FORCE—MAGNETIC INDUCTION—MAGNETIC PROPERTIES OF IRON AND STEEL—ENERGY OF THE MAGNETIC FIELD—EFFECTS OF TEMPERATURE ON MAGNETISATION—DIAMAGNETIC SUBSTANCES—PERMANENT MAGNETS

107. Magnetic Force in a Magnetic Material.—Consider a uniform field of intensity H. If a rod of soft iron be placed in this field, the iron becomes magnetised with its magnetic axis in the direction of the field.

In order to find the strength of the field at any point in the iron it is necessary to determine the strength of the field at that point due to each element of free magnetism developed on the surface of the iron, and to superpose the resultant of all these forces on the force due to the original field at that point.

Thus, to find the actual force at, say, the central point P of the rod of iron (Fig. 118), it is necessary to find the resultant of all such quantities as $\frac{\delta m}{r^2}$, where r cms. is the distance of P from the element of magnetism δm taken for the whole of the magnetism on the surface of the magnet, and to superpose this resultant on the force at P due to the original field. For the point P it is clear, by reason of symmetry, that the resultant force due to the induced magnetism of the rod will be along the axis. It is also clear that the direction of this resultant force will be directly opposed to the forces at P due to the original field. The effect, therefore, of introducing the rod into the field is to diminish the force at the point P. In order to actually calculate this force it is necessary to know what the distribution of free magnetism on the surface of the rod is, and the determination of this distribution is generally a problem of considerable difficulty.

The value of the magnetic force at every point inside the rod is therefore completely defined.

108. Magnetic Induction in a Magnetic Material.—Suppose now that an extremely narrow slit be cut across the iron bar at right angles to the direction of magnetisation as shown in Fig. 119, thus dividing the rod into two portions X and Y.

It is a matter of experimental fact that across this slit lines of force will pass (see § 75), and magnetism of opposite signs will appear on the two faces of the slit. If the slit is indefinitely narrow, the forces due to the magnetism on each face of the slit will neutralise each other at all points outside the slit.

Now the magnetism on the X face of the slit will be equal to and of opposite sign to the magnetism on the remaining sides of the X portion of the bar, since the total magnetism of the X portion is zero by § 76. Hence the same number of lines across the slit from the X face as enter the remaining sides of the X portion of the rod. Now let the material of the rod which was removed to form the slit be replaced. It is assumed that the lines still exist in the iron, or, in other words, it is assumed that each line of force outside the rod is continued right through the rod.

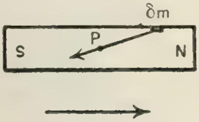

Fig. 118.

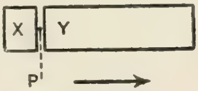

Fig. 119.

The number of lines through any given cross-section of the material can be experimentally determined, as explained in § 214.

Let H be the actual resultant intensity of the field at a point P in the iron, calculated as explained in § 107.

Let I be the intensity of magnetisation of the induced magnetism on the faces of the slit—*i.e.*, I is the strength per sq. cm. of area of the poles induced on the faces of the slit (see § 93). The magnetic lines across the slit due to the induced magnetism will be $4\pi I$ per sq. cm. (§ 89), and hence the total density of magnetic lines at a point P in the material is $H+4\pi I$.

This expression is called the **induction density** or the **induction** at the point P in the material, and is usually denoted by the letter B—that is,

$$B=H+4\pi I \text{ lines per sq. cm.}$$

In air and other non-magnetic materials the induced magnetism is zero—that is, $I=0$, and hence $B=H$.

The lines of force and lines of induction are therefore identical in non-magnetic material, and it is only in the case of materials for which I is not zero that a distinction between the lines of force and the lines of induction must be made.

It is of great importance to notice that the lines of induction are continuous lines which close on themselves. Thus, in calculations dealing with the magnetic circuit of a dynamo, the total flux of magnetic induction (or *magnetic flux*) is taken as the basis of calculation, since this flux is constant throughout the magnetic circuit whatever the nature of the media at different parts of the circuit may be (see §§ 192 and 193.) Thus—

$$\text{Total flux of magnetic induction} = B_X \times A_X,$$

where B_X is the induction density at any part of the circuit, and A_X is the cross-sectional area of that part of the circuit.

The expression connecting the induction density B and the magnetic force H to which this value of B is due at any point in a magnetic field is, as already stated—

$$B = H + 4\pi I.$$

If this is written

$$B = \mu H,$$

the factor μ is termed the **permeability** of the medium. Rewriting these expressions—

$$\mu H = H + 4\pi I,$$

$$H(\mu - 1) = 4\pi I,$$

$$I = \frac{\mu - 1}{4\pi} H = \kappa H.$$

$\kappa = \frac{\mu - 1}{4\pi}$ is termed the **susceptibility** of the medium, and gives the relationship between the intensity of the induced magnetism and the magnetic force at any point in a magnetic field.

In iron, as has been stated in § 80, the induced magnetism is in the same direction as the inducing magnetic force. In some substances, however, notably bismuth, the direction of the induced magnetism is directly opposite to that of the inducing magnetic force (see Fig. 151), and in such a case the induction is given by the expression

$$B = H - 4\pi I.$$

For bismuth $k = -\frac{1}{80000}$, whereas for iron k may have a value as high as 300 (see Figs. 135 and 136).

109. Case of a Uniformly Magnetised Iron Ring.—It is instructive at this stage to consider briefly the conditions as regards the lines of induction and the lines of force for another case which is very important in the experimental investigation of the magnetic properties of materials.

It may be shown (see § 112, also Chapter XIV., § 214) that if a coil of insulated wire be wrapped closely and uniformly round an iron ring of uniform section, and a direct current be passed through the wire, the magnetic force is the same at all points on a circle drawn in the ring, and concentric with the ring (see Fig. 120). The direction of the magnetic force is along the tangent to the circle, or, in other words, the circle is a line of force. Further, all the lines of force in the iron will be concentric circles.

Now no free magnetism will be induced on the surface of the ring, as will be clear from consideration of symmetry, there being no reason why any point on the surface should be magnetically different from any other point. The magnetic force inside the iron ring is therefore the same as it would be if the material of the ring were non-magnetic (see § 107). This magnetic force can be easily calculated from the dimensions of the circuit and the strength of the exciting current, as shown in § 112. It is for these reasons that this case is of such importance in practice.

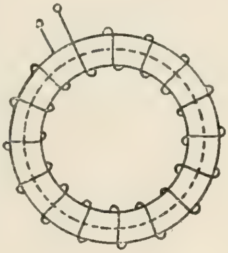

FIG. 120.

Specimens of iron for which the connection between B and H is required (see Fig. 131) are frequently made in the form of such a closed ring, and are thus free from any self-demagnetising effects due to the induced magnetism of the iron.

Such closed ring specimens are also very useful for the determination of the hysteresis losses (see § 113, Fig. 127, also § 203, Figs. 126 and 127).

Let I be the intensity of induced magnetism in the iron ring. The direction of magnetisation will be along the lines of force, the lines due to the magnetisation being thus closed concentric circles, and none leave the surface of the ring. The induction density in the ring is—

$$B = H + 4\pi I.$$

The total magnetic flux in the ring is $\Phi = B \times A$, where A sq. cms. is the cross-sectional area of the ring.

The flux may be easily measured by means of a ballistic galvanometer (§ 214), and hence the values of the induction B and the permeability μ may be found for any value of the magnetising force, H.

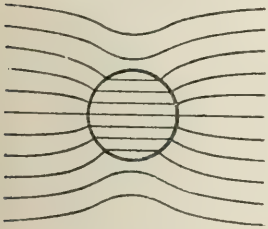

FIG. 121.

110. Self Demagnetising Force due to a Magnet.

Case I.—Suppose a sphere of magnetic permeability μ is placed in a uniform magnetic field.

It can be shown by actual calculation that the force inside the sphere will be constant at all points—*i.e.*, the magnetic lines in the sphere will be parallel and equidistant. Fig. 121 shows the distribution of the flux inside and outside such a sphere. The calculation shows that if the intensity of the field before the introduction of the sphere is H the intensity of the force in the sphere when placed in this field will be—

$$H_I = H - \frac{\mu - 1}{\mu + 2} H$$

$$= \frac{3}{\mu + 2} H.$$

If the sphere is iron, μ will generally be a very large number, and the intensity of the force at all points in the sphere will be very approximately—

$$\frac{3}{\mu} H.$$

The intensity of the induced magnetism of the sphere will be the same at every point of the sphere, and is given by—

$$I = \frac{3}{4\pi} \frac{\mu - 1}{\mu + 2} H.$$

If the sphere is of iron—*i.e.*, μ is large—this becomes

$$I = \frac{3}{4\pi} H, \text{ approximately.}$$

The value of I is thus independent of μ for all materials for which μ is large.

The magnetic force in the sphere may be written—

$$H_I = H - YI,$$

where $Y \cdot I$ is the demagnetising force, and Y is called the *demagnetising coefficient* (see also § 123).

But $$I = \kappa H_I,$$

where κ is the susceptibility (see § 108).

Therefore $$\frac{H_I}{H} = \frac{1}{1 + \kappa Y}.$$

For a sphere $$Y = \frac{4\pi}{3}, \text{ as shown above.}$$

For soft iron $$B = 10000 : H_1 = 3 :$$

$$I = \frac{B - H_1}{4\pi} = 800 \text{ approximately,}$$

and $$I = \kappa H_1,$$

or $$\kappa = \frac{800}{3} = 266,$$

$$\kappa Y = 266 \times \frac{4\pi}{3} = 1120.$$

Therefore $$H_1 = \frac{H}{1120} = 0\cdot0009\ H.$$

The magnetic force in the iron sphere is therefore extremely feeble as compared with the intensity of the original field before the iron sphere was introduced.

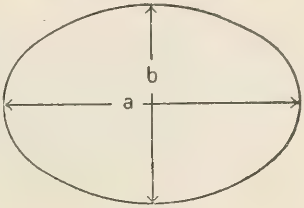

FIG. 122.

The corresponding case for electrostatic fields is obtained by putting ϵ for μ (see Figs. 76 and 77).

Case II.—Suppose an ellipsoid of revolution of axes a and b (Fig. 122) *is introduced into a uniform magnetic field in the direction of the axis a.*

It can be shown that the intensity of force in the ellipsoid will be everywhere the same. The value of the self-demagnetising coefficient Y in the formula

$$YI = H - H_I$$

is $$Y = 4\pi\frac{b^2}{a^2}\left(\log_e\frac{2b}{a} - 1\right) \text{ when } \frac{a}{b} \text{ is large.}$$

For $a = 100b$, $Y = 0\cdot0054$: for $a = 500b$, $Y = 0\cdot0003$.

If, as before, $\kappa = 266$,

$$\frac{H_I}{H} = \frac{1}{1 + \kappa Y} = \frac{1}{1 + 1\cdot43} = 0\cdot41 \text{ for } a = 100\ b,$$

and $$\frac{H_I}{H} = \frac{1}{1\cdot08} = 0\cdot926 \text{ for } a = 500\ b.$$

Hence, if a piece of iron of the form of an ellipsoid, of which the length is great compared with its breadth, is placed in a magnetic field of intensity H, the resultant intensity of the field at any point inside the iron is very approximately the same as the intensity of the original field before the introduction of the iron ellipsoid. In other words, the self-demagnetising force of the iron is very small.

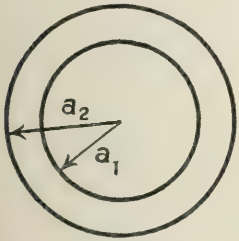

FIG. 123.

Case III.—A long straight iron wire introduced into a magnetic field which is uniform before the introduction of the wire.

Let the wire be placed so that the direction of its length coincides with the direction of the field.

Experiment shows that in such a wire the polarity is developed chiefly near the ends, and the demagnetising force at the centre is small.

Case IV.—A hollow spherical iron shell of permeability μ placed in a uniform magnetic field (Fig. 123).

If H is the intensity of the original field before the introduction of the iron shell, the intensity of the field inside the shell, when placed in this magnetic field, is—

$$H_o = \frac{H}{1 + \frac{2}{9}\frac{(\mu - 1)^2}{\mu}[1 - \beta]},$$

where $$\beta = \left(\frac{a_1}{a_2}\right)^3.$$

Thus if $\mu = 3000 : a_2 = 2a_1$

$$H_o = \frac{H}{1 + \frac{6000}{9}\left[1 - \frac{1}{8}\right]} = \frac{H}{590}.$$

An iron shield surrounding a galvanometer is sometimes used in order to protect the galvanometer from the action of external magnetic fields.

The space inside the shell is thus very effectively insulated from external magnetic fields.

With further reference to the self-demagnetising effects of magnets, see § 123.

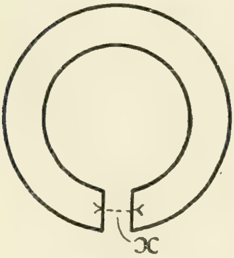

FIG. 124.

111. Magnetic Force between given Magnetic Masses in a Medium of Permeability μ.—Consider a ring of iron with a very narrow radial slot (Fig. 124), and let the faces of the slot be magnetised uniformly with an intensity +I and – I respectively.

There will then be a flux across the radial slot.

The force at a point in the slot will be

$$H = 4\pi I,$$

the induction density at P will also be

$$B = 4\pi I,$$

and this will also be the value of B throughout the ring.

Now suppose the iron ring immersed in a medium of permeability μ_o—say oxygen, for example (see § 118)—and let the intensity I be kept constant.

The medium in the slot will become magnetised (Fig. 125), and suppose the intensity of magnetisation thus produced be I_o.

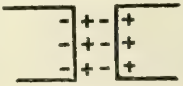

FIG. 125.

It is clear that the magnetism of the medium will oppose the magnetism on the faces of the slot, so that the effective intensity of magnetism at the face of the slot will be $I - I_o$. The force at the point P in the slot will therefore be

$$H_o = 4\pi (I - I_o).$$

The value of B in the iron ring, however, will not be altered, since I remains the same. Further, since B is continuous, the value of B in the slot is the same when the ring is immersed in an oxygen medium, as when in air.

Hence $$\mu_\circ=\frac{B}{H_\circ}=\frac{I}{I-I_\circ}.$$

Therefore $$\frac{\text{Magnetic force in oxygen}}{\text{Magnetic force in air}}=\frac{H_\circ}{H}$$
$$=\frac{I-I_\circ}{I}$$
$$=\frac{1}{\mu_\circ}.$$

The force between two magnetic masses in a medium of permeability $\mu_\circ$ is thus $\frac{1}{\mu_\circ}$ times the force between the same two magnetic masses when in air. This agrees with the statement of Coulomb's Law given in § 84.

112. Relationship between the Magnetic Induction B and the Magnetic Force H in Iron.—Consider a coil of insulated wire regularly wound in the form of a ring, and supplied with direct current, as previously referred to in § 109.

Suppose in the first place that no magnetic material is present in the coil. For each value of the current flowing in the coil a definite value of the magnetic force at points in the coil will be developed. The intensity of the magnetic force will be given by the expression—

$$H=\frac{4\pi}{10}\frac{iw}{l},$$

where i is the current in amperes, w is the number of turns in the coil, and l cms. is the length of the mean circumference shown dotted in Fig. 126. For proof of this see § 192.

If such a coil is wound on an iron ring, the iron becomes magnetised, but no free magnetism is developed in the iron—*i.e.*, no magnetic poles appear, and consequently there is no self-demagnetising effect. There is, therefore, no reason why the magnetic force should differ, whether the iron is present or removed, since there is no free magnetism developed in either case. The magnetic force for a given exciting current in the coil is therefore the same in the space inside the coil, whether the iron is there or not; and, as stated in § 109, the magnitude of the force is the same at all points on a circle such as the dotted circle in Fig. 126.

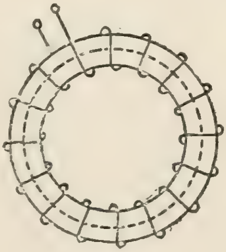

FIG. 126.

Further, as can quite easily be demonstrated, if a narrow radial slit be cut in the iron, lines of force will cross the slit, and wherever the slit is made the flux crossing it will be the same for a given width of slit. It is then assumed that a corresponding magnetic flux exists in the iron when the slit is filled up (see also § 108).

A proof that the flux does actually exist in the closed iron ring when current flows in the exciting coil is given by the fact that if a second or "search coil" be wound round the exciting coil and the current in the latter be broken, an exciting magnetic flux is induced in the second coil, and, in fact, the throw given on a ballistic galvanometer connected to the search coil is a measure of the flux. Suppose for a given value of H, as calculated from the formula given above, the corresponding flux density in the iron is B.

Then $$B=H+4\pi I,$$

where I is the intensity of magnetisation developed in the iron (see § 108). The permeability is

$$\mu=\frac{B}{H}.$$

If corresponding values of B and H be plotted using rectangular co-ordinates, the following general results will be obtained (see Fig. 127).

Assume that the iron is obtained in a completely demagnetised state—*i.e.*, $H=o$: $B=o$. The curve starts at the origin O, and as the value of H is increased B also increases, as shown by the curve O*a*, and for low values of H, B will increase in almost direct proportion with the increase of H. As H increases still further, the rate of increase of B becomes greater, and the curve *ab* is obtained. Further increase of H results in an increase of B, but at a less rate, and the curve bends towards the axis of H, as shown by *bc*. The curve O*abc* is called the **magnetisation curve** of the iron (see also §§ 114 and 115).

113. Hysteresis.—Suppose in Fig. 127 that after the point *c* on the magnetisation curve has been reached, the value of H is gradually reduced. The curve now obtained connecting B and H does not follow the previous curve *cba*O, but lies higher, as shown by *cd*, so that when H has become reduced to zero the value of B is given by O*d*. This value O*d* is the **remanent magnetism** or **remanence** of the iron.

Since $$B=4\pi I+H,$$

and for the point *d* the value of H is zero, it follows that $B_{rem.}=4\pi I_{rem.}$

As H is still further reduced—*i.e.*, becomes negative—the corresponding values of B are given by the curve *de*, so that when B becomes zero H has the negative value *oe*. This value of H is the force necessary to reduce the magnetic flux in the iron to zero, and is equal and opposite to the **coercive force.**

As H is still further increased in the negative direction B becomes negative, and the curve *ef* is obtained, such that when the negative value of H—viz., *og*—is the same as the maximum positive value of H—viz. *oh*—the negative value of B is equal to the positive value of B—viz., $gf = hc$.

If the negative value of H be reduced the curve *fl* is obtained, so that when H is zero B has the negative value *ol*.

When H is increased in the positive direction, the curve *lc* is obtained, so that when H again reaches its original maximum positive value, the value of B also is the same as its former maximum positive value, and the curve meets the previous curve at *c*. In this way a closed loop *cdeflc* is obtained, which is known as the **hysteresis loop.** The word "hysteresis" means the "lagging" of the magnetic induction behind the magnetic force when the force is varied.

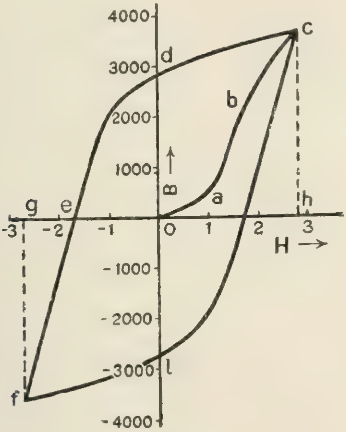

Fig. 127.

As will be shown later (§ 116), the area of the hysteresis loop represents a definite amount of energy lost in performing the cycle of operations described, which loss is termed the **hysteresis loss.** The energy represented by the hysteresis loop is dissipated in heat.

Steinmetz deduced from a large number of tests that the hysteresis loss per cycle is closely proportional to the 1·6th power of the maximum induction density reached during the cycle, and that this relationship holds over a range of maximum induction densities from about $B_{max} = 1000$ to B_{max}

$=13000$. Thus the energy loss due to hysteresis may be written—

$$
\begin{aligned}
U_h &= \eta\, B_{max}^{1\cdot6} \text{ ergs per cubic cm. per cycle,}\\
&= \eta\, B_{max}^{1\cdot6} \times 10^{-7} \text{ joules per c.c. per cycle,}\\
&= 58\, \eta\, B_{max}^{1\cdot6} \times 10^{-7} \text{ joules per lb. per cycle.}
\end{aligned}
$$

The factor η is known as the *hysteresis constant* of the iron. The value of η depends upon the quality of the iron, and may have widely different values for different samples. For good transformer iron η may be taken as 0·0013. In Fig. 128 is shown the connection between energy loss in joules per lb. per cycle and the values for B_{max} for $\eta=0\cdot0013$.

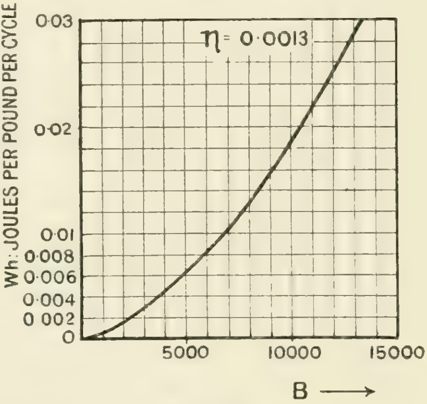

FIG. 128.

Some experiments recently carried out in America on iron, iron-silicon alloys, and iron-aluminium alloys, melted *in vacuo* in an electric furnace, gave extraordinary results as regards low hysteresis loss and high maximum permeability.

For electrolytic iron with a trace of carbon the maximum value of the permeability was found to be 19000, whereas the maximum permeability of good armature iron is only about 4000 (see Fig. 135). The hysteresis loss per cycle for $B_{max}=10000$ was 813 ergs per c.cm. per cycle, which corresponds to a hysteresis constant $\eta=0\cdot00032$.

For iron with no carbon but with about 3·4 % of silicon, the maximum permeability obtained was over 60000, and the hysteresis loss about 300 ergs per c.cm. per cycle for $B_{max}=10000$. This corresponds to a hysteresis constant $\eta=0\cdot00012$. The specific resistance (§ 142) of this iron was about $48\cdot5\times10^{-6}$ ohms per cm. cube.

For iron with 0·4 % aluminium annealed at 1100° C., the maximum permeability was 35000, and the hysteresis loss for $B_{max}=10000$ was 450 ergs per c.cm. per cycle, *i.e.* $\eta=0\cdot00018$.

In the following table are given representative values of the hysteresis constant for iron, steel, and cast cobalt.

Material.	*Hysteresis Constant* η.
Hard tungsten steel, as used for permanent magnets	0·052
Hard carbon steel	0·045
Cast cobalt	0·01
Good armature iron	0·0013
Silicon iron ("stalloy")	0·00085
Iron melted *in vacuo* and containing a trace of carbon	0·0003
Iron melted *in vacuo* and containing a small percentage of silicon	0·0001

It has been pointed out that the Steinmetz formula for hysteresis loss is applicable for a range of B_{max} from about 1000 to 13000 lines per square cm. It does not hold, however, for values of B considerably less than 1000 lines per sq. cm., and therefore such practical cases as the hysteresis loss in telephone transformers, in which the value of B_{max} may be of the order of 1 line per sq. cm. or less, cannot be dealt with by this formula. From experiments on a ring of soft iron Baur found that for values of H up to about 0·4 the relationship between B and H could be expressed by the formula—

$$B = 183\ H + 1382\ H^2,$$

from which the permeability is

$$\mu = \frac{B}{H} = 183 + 1382\ H.$$

It follows, therefore, that for exceedingly small values of H the value of the permeability μ should be very approximately constant and equal to 183.

Lord Rayleigh investigated this matter by means of experiments on unannealed Swedish iron wire, and found that for values of H less than 0·04 the permeability had the constant value of about 100. There was, therefore, no remanent magnetism and no hysteresis loss when the magnetisation was carried through a cycle in which the maximum value of H was less than 0·04. For higher values of H this result did not hold.

For the alloyed iron known as "stalloy" (see § 115), the hysteresis loss has been given as proportional to $B_{max}^{2\cdot69}$ for values of B_{max} in the neighbourhood of 1 line per sq. cm. (see Wilson, Winson, and O'Dell, *Proc. R. S.*, "A," vol. lxxx., 1908).

114. The Molecular Theory of Magnetism.—The fundamental facts of magnetisation may be satisfactorily accounted for on the assumption that the molecules of iron and steel are permanent magnets capable of being turned round their centres. This hypothesis was first developed by Weber, and, briefly stated, the conception is as follows:

In an unmagnetised bar of iron or steel the molecular magnets are so irregularly arranged that they mutually neutralise each other's external field. When the bar is subject to a magnetising force some of the molecules are turned so that their magnetic axes lie in the direction of the force. As the intensity of the magnetising force increases, more and more molecules set in the direction of the force, and the bar becomes correspondingly more strongly magnetised. There is, however, a limit to the magnetisation of the bar, and this limit is reached when all the molecules are arranged in the direction of the magnetising force, when the iron is said to be *saturated.* This statement of the theory accounts for the observed facts to a certain extent, but it does not account for the fact that there is some internal constraint in the iron which prevents all the molecular magnets coming into line with the magnetising force until this force has reached a high value. The theory also does not account for the internal constraint which results in the remanent magnetism when the magnetising force is reduced to zero—in other words, the phenomenon of **hysteresis** is not explained.

Ewing, as the result of his experiments, has developed the theory, and accounts for the constraint as being due to the mutual magnetic action between the molecular magnets as follows:

When a piece of iron is unmagnetised the molecules are arranged in groups, each group forming a stable magnetic system and producing no external magnetic action. When such a group is subject to a small magnetising force H, each member of the group becomes slightly deflected, but the group still maintains its general formation. When the small magnetising force is removed each member returns to its original position and the external action of the group disappears. These results correspond to the initial stage *oa* of the magnetisation curve in Fig. 129. For this first stage the magnetic induction is very nearly proportional to the magnetising force, and there is no appreciable hysteresis loss when the magnetisation is carried through a cycle of which the maximum value of the induction density is within the limits of this first stage, because the hysteresis loop shrinks to a straight line (see § 113). Moreover, during this first stage the rate of increase of B with H is relatively small.

If the magnetising force H is now still further increased, the second stage of magnetisation is reached. The individual components of each group of molecules become still further deflected, and this increased deflection is sufficient to break up some of the groups. The molecules of these ruptured groups then rearrange themselves under the magnetic forces, and form new groups in which the molecular magnets are more nearly in line with the magnetising force. As H continues to increase, more groups are broken up in this way, and the part of the magnetisation curve *ab* (Fig. 129) is obtained in which B increases rapidly with H. This action continues until the condition of *saturation* is approached, the molecules become arranged in their final grouping, and the third stage of magnetisation is reached.

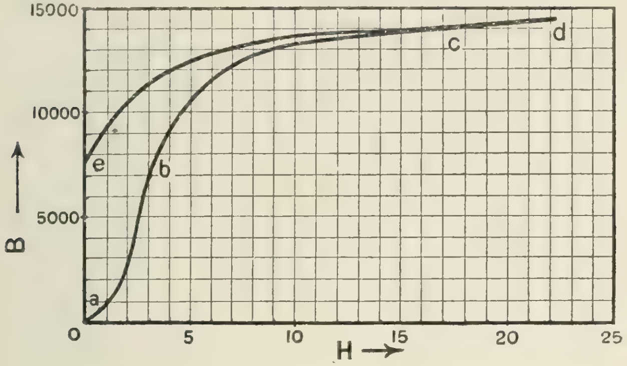

FIG. 129.

Still further increase of H causes the molecular magnets of the new groups to set more and more completely in line with the force until eventually the molecular magnets are all arranged in line with the magnetising force, and the iron is *saturated*. This stage about corresponds to the part of the curve *abcd* in Fig. 129 to the right of the ordinate H=10.

Now $$B=4\pi I+H,$$

and when saturation is reached I has attained its maximum value, and the expression becomes—

$$B=\text{constant}+H.$$

The maximum value of I for pure iron is about 1680, so that for values of H corresponding to saturation values (see also p. 147),

$$B=4\pi\times1680+H=21000+H.$$

It is to be observed that these three stages are not sharply defined, but that the consecutive stages gradually merge into each other, there being a range over which the phenomena of both stages 1 and 2 act, and a range over which the phenomena of both stages 2 and 3 act.

If after reaching saturation the magnetic force is gradually reduced, the magnetisation curve will coincide with the original curve in so far as the reduction of H corresponds to a change in deflection of the individual molecules of a group, but not to the rupture of a group.

As the magnetising force is still further reduced the molecular groups begin to rupture, but owing to the forces between the components of the group there is a constraint which prevents the magnetism of the iron being reduced proportionally to the reduction of the magnetising force, so that when H is reduced to zero there is a considerable amount of residual magnetism in the iron corresponding to the point *e* in Fig. 129. In other words, the magnetic induction *lags* behind the magnetising force, so that an increase or decrease of H is not accompanied by a proportional increase or decrease of B, and this is known as the *hysteresis* effect. If it be assumed that mechanical vibration of the iron results in a greater freedom of movement of the molecules, the fact that such mechanical vibration reduces the amount of remanent magnetism is also accounted for, and similarly the fact that a given magnetising force produces a greater degree of magnetisation in a bar which is being hammered than one which is not so treated.

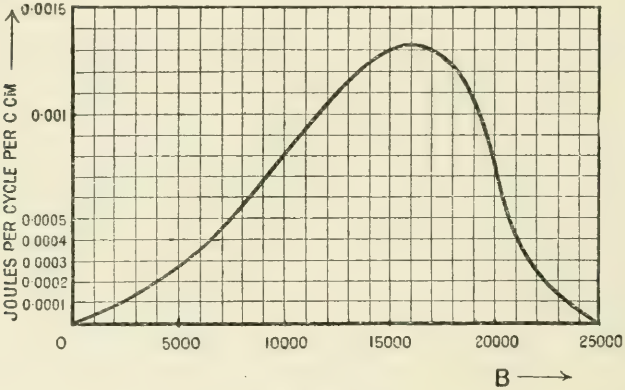

Fig. 130.

The hysteresis effect entails a loss of energy dissipated in heat, because when each molecular group is ruptured the individual members are set vibrating, the corresponding kinetic energy being eventually transformed into heat.

In Ewing's experiments all these effects were exhibited by means of a model, in which a number of small compass needles were pivoted on a fixed base and were intended to represent the molecular magnets of a piece of iron (see "Magnetic Induction in Iron and Other Metals," J. A. Ewing).

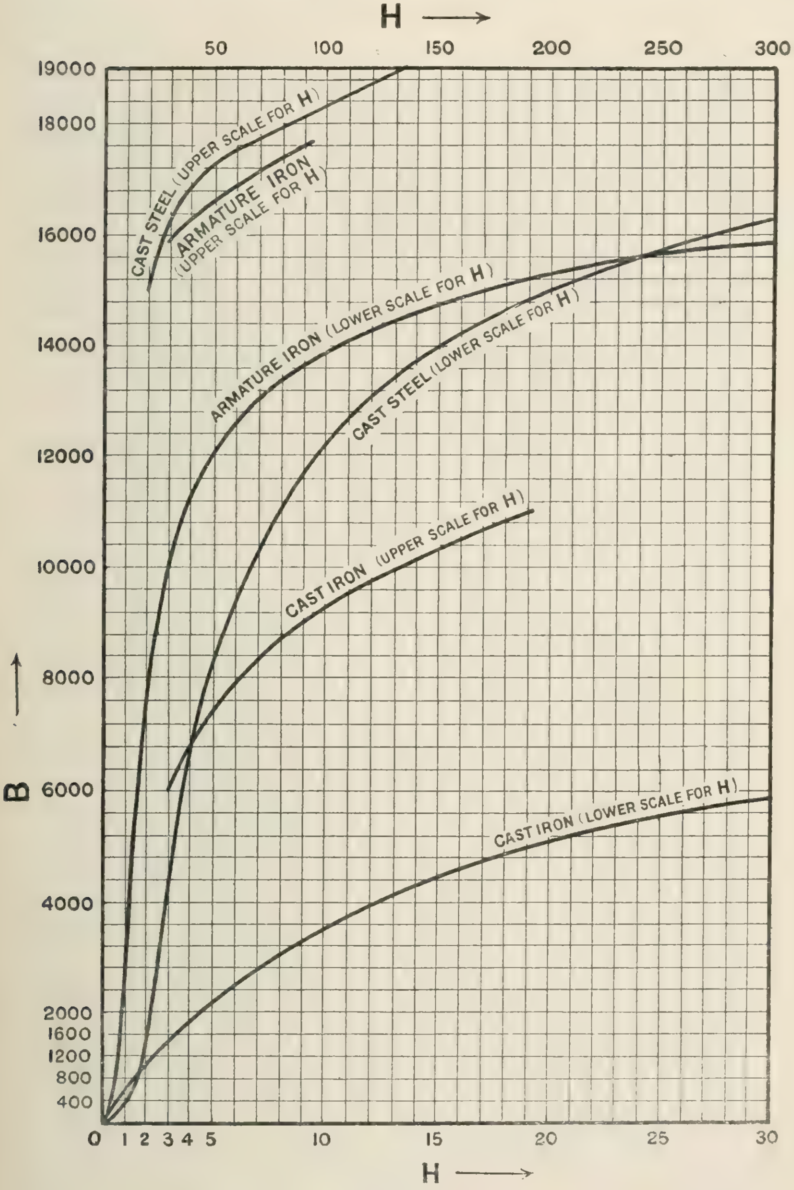

FIG. 131.

A remarkable deduction from the molecular theory of magnetism was made by Swinburne, who predicted that when iron is rotated in an extremely strong magnetic field hysteresis will disappear, since the molecular magnets will be held in line by the magnetising force throughout the cycle, and consequently there will be no dissipation of energy due to rupturing of groups of the molecular magnets. This conclusion was experimentally verified, and in Fig. 130 is shown the curve of hysteresis loss per cycle per c.cm. due to rotating sheet-iron in a magnetic field (Beattie and Clinker, *Electrician*, vol. xxxvii., p. 723). It will be seen that at an induction density of B about 16000 the hysteresis loss per cycle is a maximum, and for higher values of B diminishes, and eventually disappears for B about 25000.

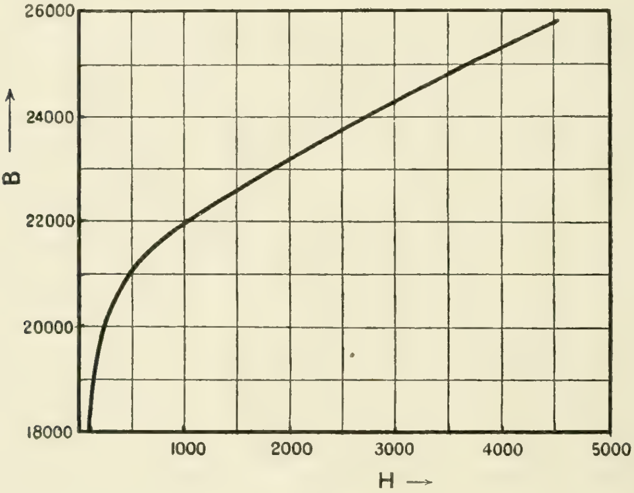

FIG. 132.

115. Magnetisation and Permeability Curves for Iron and Steel.—In Fig. 131 are shown the magnetisation (B-H) curves for good cast-iron, for dynamo steel castings (Edgar Allen's, " Imperial " brand), and armature iron stampings (J. Lysaght's brand). The curve for each material is drawn in two parts, of which the lower part refers to the bottom scale of H and the upper part to the top scale.

In Fig. 132 is shown a further extension of the B-H curve for armature iron stampings for very high values of the induction density up to and beyond saturation. It was pointed out in § 114 that saturation is reached when the value of the intensity of magnetisation I is a maximum, and if H is increased beyond the value corresponding to I_{max}, the B-H curve becomes a straight line represented by the equation—

$$\begin{aligned} B &= 4\pi I_{max} + H \\ &= \text{constant} + H. \end{aligned}$$

In Fig. 132 the curve beyond B=25000: H=3720 is a straight line, and hence—

$$4\pi I_{max}=B-H=25000-3720=21280;$$

or

$$I_{max}=1690,$$

which is about the value given in § 114.

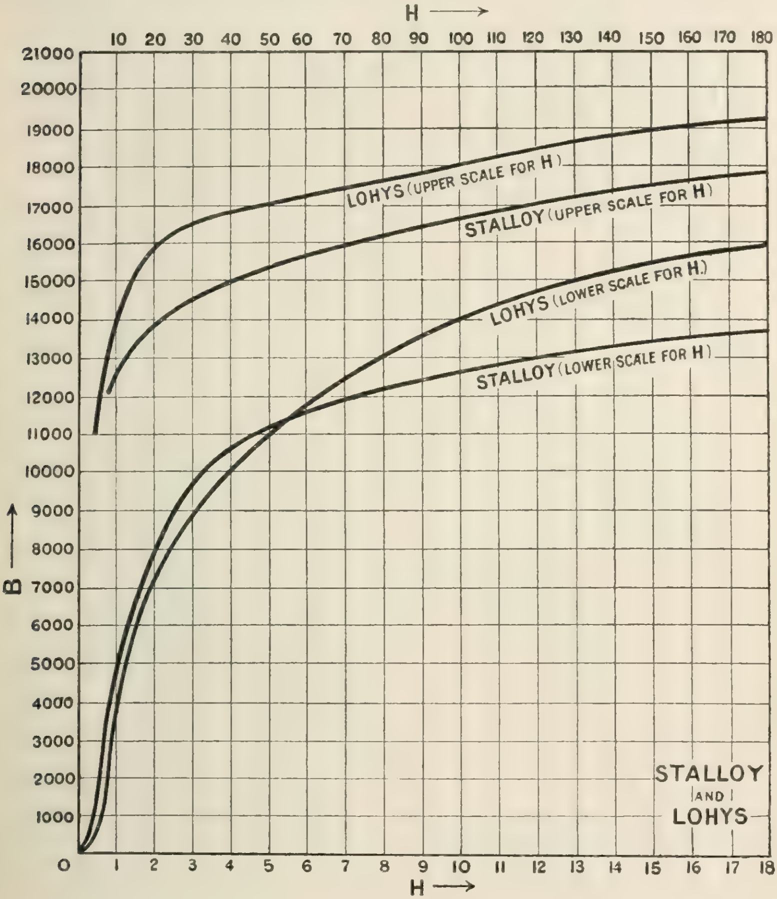

FIG. 133.

It is interesting to observe that in Fig. 130 the hysteresis loss, due to the rotation of iron in a magnetic field, disappears when B=25000, and from what has been said in § 114 this disappearance should occur when the iron is saturation, and hence the value of B for saturation of the iron is the same when deduced from either Fig. 132 or Fig. 130.

Armature iron stampings are very nearly pure iron, and it has been found that no material has a greater value of I_{max} than pure iron. The addition of small percentages of carbon to the iron has the effect of reducing the value of I_{max} by an amount proportional to the amount of carbon.

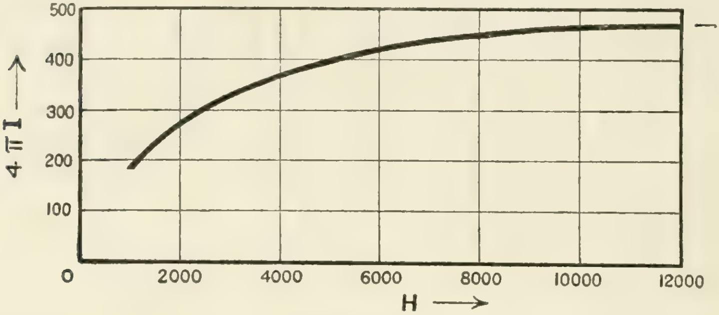

Fig. 134.

Some years ago certain alloys of iron were introduced for electrical machinery and apparatus, the characteristic properties of the alloys being relatively very low hysteresis and eddy current losses (see §§ 113 and 274). One of the best known of these alloys is "stalloy" (containing about 3·4 per cent. silicon), produced by Messrs. Joseph Sankey, and the magnetisation curve for this material is given in Fig. 133. In Fig. 133

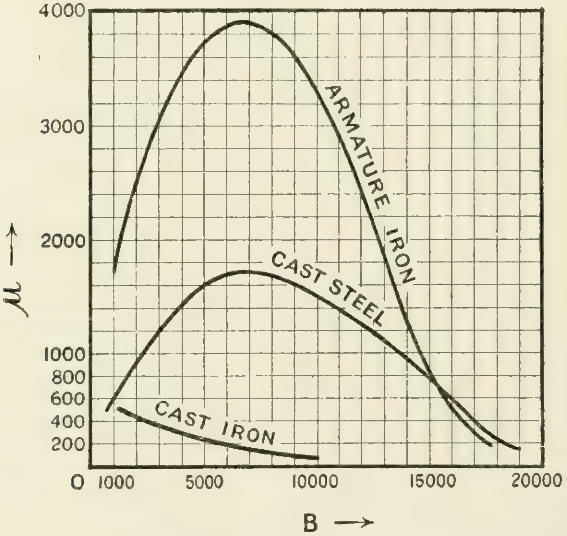

Fig. 135.

the magnetisation curve for "lohys" is also shown. This material is a good quality armature iron, also produced by Messrs. Sankey, who have supplied the data for Figs. 133, 136, 137, and 138. From a comparison of the curves in Fig. 136 it will be seen that for low and medium values of B "stalloy" has a higher permeability than armature iron (compare also Figs. 131, 133, 135, 136). For high values of B "stalloy" requires a greater magnetising force than armature iron. Hence, for apparatus which normally works with medium values of the induction density, "stalloy" is superior to armature iron, owing to its lower iron losses, whereas if the apparatus is required to work at the highest possible flux density and the losses are of secondary consideration, armature iron stampings are superior. The cost of "stalloy" is, however, greater than that of armature iron.

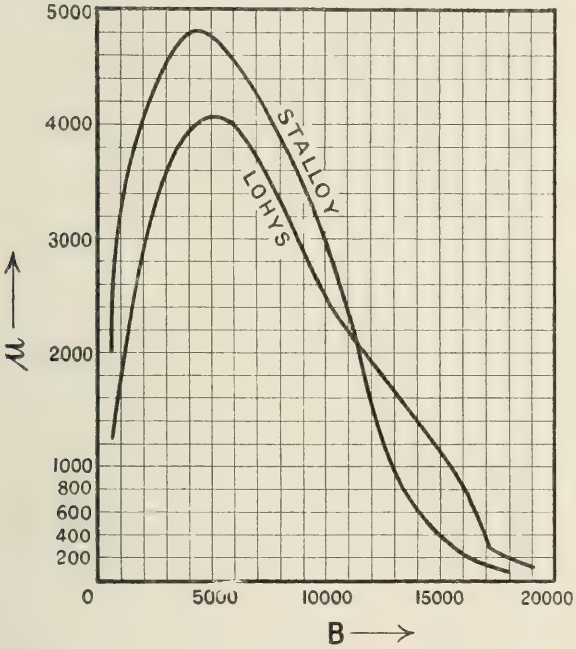

Fig. 136.

Messrs. Hadfield have succeeded in producing a steel of composition such that it is almost non-magnetic, and for certain purposes this is very desirable. A feature of this steel is that it is intensely hard, and as it is practically impossible to machine it, the only practicable operation to which it can be submitted, apart from forging, is grinding.

In Fig. 134 is given a curve showing the connection between $4\pi I$ and H for such steel having 1·66 per cent. carbon and 11·53 per cent. manganese and heat treated. Owing to the exceedingly small maximum value of I obtainable with this steel the values of $4\pi I$ have been plotted for clearness instead of B. It will be seen from the curve that the maximum permeability of this steel is only about 1·2 and I_{max} about 38 (see Hadfield and Hopkinson, *Journal I.E.E.*, vol. xlvi.).

In Fig. 135 are shown permeability curves deduced from the curves of Fig. 131, the values of the induction B being plotted horizontally, and the corresponding values of μ plotted vertically.

In Fig. 136 the permeability curves for "stalloy" and "lohys" are shown, these curves having been deduced from the corresponding magnetisation curves of Fig. 133.

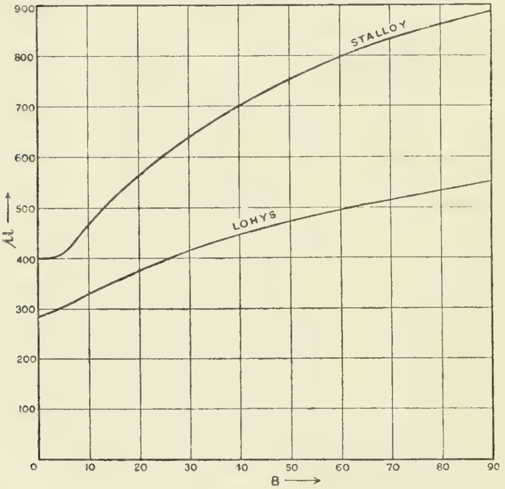

FIG. 137.

For such apparatus as telephone transformers for which extremely small values of the flux density are used, it is necessary to know the value of the permeability at these low densities. In Fig. 137 are shown permeability curves for "stalloy" and "lohys" for values of the induction density B from 0 to 90 lines per sq. cm.

In Fig. 138 the permeability curves of "stalloy" and "lohys" are shown for values of B from 0 to 2000 lines per sq. cm. (see also Chapter XX., § 274).

116. Energy Involved in the Magnetisation of Iron.—Suppose an iron bar of cross-sectional area S sq. cms. is brought into a uniform field, and that the bar is in consequence magnetised with a uniform intensity of magnetisation I as shown in Fig. 139. In the first place it is assumed that the actual resultant field strength H is uniform throughout the bar.

Now let the field intensity H of the bar be gradually reduced, the intensity of magnetisation I being correspondingly reduced so that the relationship between H and I is given by some such curve as PR (Fig. 140).

In proceeding from the condition given by *p* to that given by *q* (Fig. 140), let the intensity of magnetisation be reduced from I (*i.e.*, O*s*) to I – δI (*i.e.*, O*r*), and let the corresponding reduction of the field intensity be from H (*i.e.*, *sp*) to H – δH (*i.e.*, *rq*). The energy recovered in demagnetising the bar by the amount δI will be equal to the work done in moving the quantity of magnetism SδI from the end C to the end D of the bar, the intensity of the field being reduced from H to H – δH during the transfer.

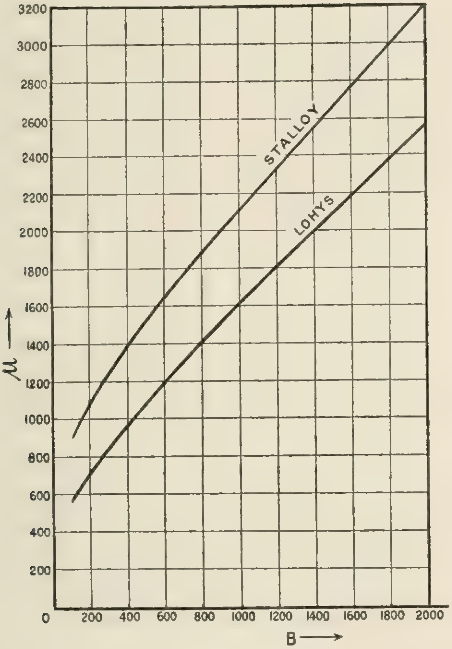

FIG. 138.

The work done during the transfer is therefore intermediate between *l*HSδI and *l*(H – δH)SδI.

Since δI and δH are small quantities, the product δIδH may be neglected, and hence the energy recovered in demagnetising the bar by the amount δI is—

$$lHS\delta I \text{ ergs.}$$

But this expression is equal to the area of the strip *pqrs* (Fig. 140) multiplied by the volume S*l* c.cms. of the bar. It follows, therefore, that in demagnetising the bar from the position *p* to the position *q* (Fig. 140) the energy recovered per c.cm. of the bar is given by the area *pqrs*.

Similarly, in proceeding from the position P to the position R (Fig. 140), the energy recovered per c.cm. of the bar is given by the area *f*PR*g*.

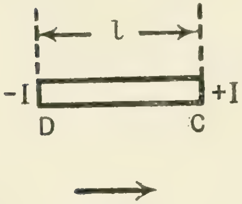

FIG. 139.

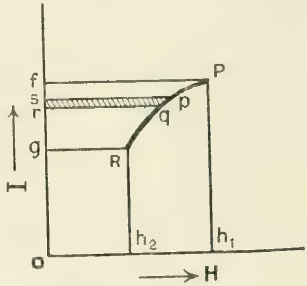

FIG. 140.

If the corresponding values of H and I are not uniform for the whole of the bar it is necessary to consider the demagnetisation of each elementary volume and the appropriate curve such as PR obtained for each element of volume.

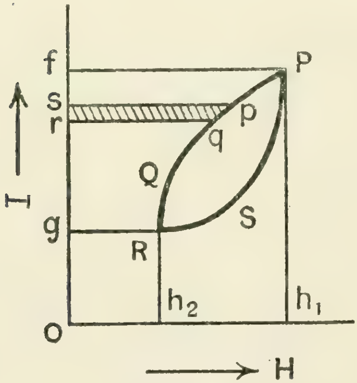

FIG. 141.

Now suppose that the magnetic state of a piece of iron initially corresponds to the conditions defined by the point P (Fig. 141). If the intensity of magnetisation I be reduced from P to R according to the curve PQR, it follows, from what has been said already, that the energy recovered per c.cm. of the iron in demagnetising it from $I = Of$ to $I = Og$ (Fig. 141) is given by the area *f*PQR*g*.

Next let the bar be again magnetised to the condition P, and suppose the connection being I and H during the process of magnetisation is given by the curve RSP. It may be shown by reasoning similar to that used in connection with Fig. 140 that the energy absorbed per c.cm. in magnetising the iron according to the curve RSP is given by the area *g*RSP*f*. Hence in performing a complete cycle of magnetisation from the condition P along the curves PQRS and back to P, *the energy absorbed per c.cm. of the iron is given by the area of the loop PQRSP.*

That is to say, the energy absorbed in performing the cycle is given by

$$\int_0 \mathrm{H}d\mathrm{I} \text{ ergs per c.cm. of the iron,}$$

the integration being performed over the complete cycle.

This energy is dissipated in heat and is the *hysteresis loss* in the iron (see § 113).

Instead of expressing the energy loss per cycle in terms of H and I, it may be expressed in terms of H and B.

Thus
$$\mathrm{B}=4\pi\mathrm{I}+\mathrm{H};$$
$$\mathrm{I}=\frac{\mathrm{B}-\mathrm{H}}{4\pi}.$$

Energy loss is—
$$\int_0 \mathrm{H}d\mathrm{I}$$
$$=\frac{1}{4\pi}\int_0\left[\mathrm{H}d\mathrm{B}-\mathrm{H}d\mathrm{H}\right] \text{ ergs per c.cm. of the iron.}$$

But the value of $\int \mathrm{H}d\mathrm{H}$ over a complete cycle is zero, hence: *The energy dissipated in heat in ergs per c.cm. per cycle of magnetisation is* $\frac{1}{4\pi}$ *times the area of the loop formed by the B-H curve.*

Example I.—An important practical case is that in which a ring of iron is magnetised by means of a coil of insulated wire wound on the ring (Fig. 142), and excited by continuous currents, as in § 109.

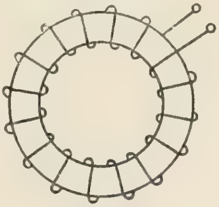

Fig. 142.

Let H be the value of the magnetic intensity in the ring, and B the corresponding value of the induction density.

Suppose the ring is initially unmagnetised, and then magnetised as shown by the curve OPQ (Fig. 143), the magnetisation being eventually carried through the cycle QRSTQ.

The energy dissipated in heat is given by—

$$\left(\frac{1}{4\pi}\text{ area of loop QRSTQ}\right) \text{ ergs per c.cm.}$$

Example II.—As another example, suppose iron is magnetised from a position defined by the point P to that defined by the point Q (Fig. 143).

The energy absorbed in *magnetising the iron* from the condition P to the condition Q is—

$$\int_{I_1}^{I_2} \mathrm{H}\,d\mathrm{I} \text{ ergs per c.cm.}$$

The energy necessary to *increase the intensity* of the field from $H_1 = B_1P$ to $H_2 = B_2Q$ is shown in § 117 to be—

$$\frac{1}{4\pi}\int_{H_1}^{H_2} \mathrm{H}\,d\mathrm{H} \text{ ergs per c.cm.,}$$

Hence the total energy absorbed in proceeding from the condition P to the condition Q is—

$$\int_{I_1}^{I_2} \mathrm{H}\,d\mathrm{I} + \frac{1}{4\pi}\int_{H_1}^{H_2} \mathrm{H}\,d\mathrm{H} \text{ ergs per c.cm.}$$

$$= \frac{1}{4\pi}\int_{B_1}^{B_2} \mathrm{H}\,d\mathrm{B} - \frac{1}{4\pi}\int_{H_1}^{H_2} \mathrm{H}\,d\mathrm{H} + \frac{1}{4\pi}\int_{H_1}^{H_2} \mathrm{H}\,d\mathrm{H} \text{ ergs per c.cm.}$$

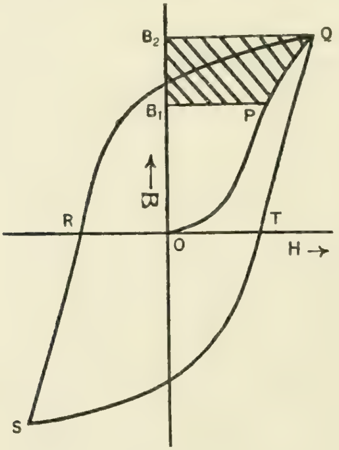

Fig. 143.

Since $\delta \mathrm{I} = \dfrac{\delta \mathrm{B} - \delta \mathrm{H}}{4\pi}$

That is, the total energy absorbed in proceeding from P to Q is—

$$\frac{1}{4\pi}\int_{B_1}^{B_2} \mathrm{H}\,d\mathrm{B} \text{ ergs per c.cm.,}$$

and this is the area of the shaded portion B_1PQB_2 (Fig. 143) divided by 4π.

The above expression for the energy may be written—

$$\frac{1}{4\pi}\int_{B_1}^{B_2} H\,dB = \frac{1}{4\pi}\int_{B_1}^{B_2} \frac{B}{\mu}\,dB—$$

If the magnitude of μ may be assumed constant over the range of B from B_1 to B_2 the energy supplied in increasing the induction from B_1 to B_2, and correspondingly increasing the field intensity from H_1 to H_2, may be written—

$$\frac{1}{4\pi}\,\frac{1}{\mu}\int_{B_1}^{B_2} B\,dB = \frac{B_2^2 - B_1^2}{8\pi\mu} = \mu\frac{H_2^2 - H_1^2}{8\pi} \text{ ergs per c.cm.}$$

For example, if $H_1 = o$: $B_1 = o$, and the iron is magnetised to the condition Q by the process defined by the curve OPQ, the total energy actually absorbed in ergs per c.cm. to establish field is given by the area $OPQB_2O$ divided by 4π.

If, however, the value of μ for the iron is assumed to be constant throughout the process, the total energy may be represented, as shown above, by the formula—

$$\frac{B_2^2}{8\pi\mu} \text{ ergs per c.cm.}$$

—that is,

$$\frac{1}{4\pi}\,\frac{B_2 H_2}{2} \text{ ergs per c.cm.,}$$

which is $\frac{1}{4\pi}$ times the area of the triangle formed by the straight line joining OQ and by the sides OB_2 and B_2Q.

Hence the error involved in assuming μ to be constant is the error involved in replacing the curve OPQ by the straight line joining OQ.

117. The Energy of the Magnetic Field.—It was shown in § 54 that the energy of an electric field in air could be accounted for if it be assumed that the energy stored per cubic centimetre is—

$$\frac{F^2}{8\pi} \text{ ergs,}$$

where F is the intensity of the electric field at the place considered.

If a unit area be taken at right angles to the direction of F it is clear that F unit tubes of force will cross this area, and hence the energy to b attributed to each unit tube of force is—

$$\frac{F}{8\pi} \text{ ergs per cm. length of the tube.}$$

Comparing the expression

$$\frac{F^2}{8\pi} \text{ ergs per c.cm.}$$

with the expression for the energy stored in a stretched elastic rod—viz.:

$$\text{Energy stored} = \frac{1}{2} \text{ (intensity of force} \times \text{strain) per unit length of rod,}$$

it was seen that the quantity $\frac{F}{4\pi}$ is analogous to mechanical strain.

In the case of a magnetic field, since the force between magnetic masses in air is defined by precisely the same law as the force between electric charges in air, the immediate deduction may be made that the energy of a magnetic field in air may be accounted for by assuming that

$$\frac{H^2}{8\pi} \text{ ergs per c.cm.}$$

are stored.

Since across unit area taken at right angles to the direction of the force H unit tubes of force pass, the energy stored in the field in air is—

$$\frac{H}{8\pi} \text{ ergs per cm. length of unit tube.}$$

Now consider the case of a ring permanent magnet with a radial air gap, and let B be the magnetic induction in the gap. The intensity of magnetic force in the gap will also be equal to B, and the energy stored per cm. length of unit tube in the gap will be—

$$\frac{B}{8\pi} \text{ ergs per cm. length of unit tube;}$$

or, since there are B unit tubes per sq. cm., the energy stored per unit volume of the gap will be—

$$B\frac{B}{8\pi} = \frac{B^2}{8\pi} \text{ ergs per c.cm.}$$

Now suppose the magnet to be immersed in a magnetisable medium such as oxygen (see § 121), of which the permeability is μ_0, and suppose that B unit tubes are maintained across the gap when thus immersed. The intensity of the magnetic force in the gap will now be $\frac{B}{\mu_0}$ (§ 111). The energy stored per unit tube in the gap will be $\frac{B}{8\pi\mu_0}$ ergs per cm. length of unit tube, or since there are B tubes per sq. cm. the energy stored per unit volume is—

$$\frac{B^2}{8\pi\mu_0} \text{ ergs per c.cm.}$$

The general expression for the energy of a magnetic field may be written—

$$\frac{B^2}{8\pi\mu} = \frac{1}{8}\frac{B}{\pi\mu} B \text{ ergs per c.cm.}$$

$$= \frac{1}{2} H \frac{B}{4\pi} \text{ ergs per c.cm.,}$$

where H is the magnetic intensity and B the magnetic induction in the field.

This expression for the energy absorbed in establishing the field only holds if μ is constant (§ 116), and can only be considered as " stored " in the sense of being completely recoverable, if there is no hysteresis.

It is of interest to estimate from the formula for the energy of the magnetic field, viz.:

$$\frac{B \times H}{8\pi} = \frac{B^2}{8\pi\mu} = \frac{\mu H^2}{8\pi} \text{ ergs per c.cm.,}$$

what the maximum amount of energy is which can be stored in non-magnetic materials.

Intense magnetic fields are obtained by means of electromagnets, and the practical limit to the value of H is set by the heating of the coils of the electro-magnet due to the exciting current. The maximum value of H practically obtainable by normal means is thus about 24000 lines per sq. cm.

The maximum amount of magnetic energy which it is practicable to store in non-magnetic materials is therefore about—

$$\frac{H^2}{8\pi} = \frac{(24000)^2}{8\pi} = 23 \times 10^6 \text{ ergs per c.cm;}$$

or 28 ft.-lbs. per c. inch.

This quantity should be compared with corresponding data given for the case of an electric field (see Section I., § 57).

Recently Professor K. Onnes has made experiments with the exciting coil kept at a very low temperature (see p. 199), and in this way has obtained a value of $\dot{H}=100000$ in air. The magnetic energy stored in the air in this case is—

$$\frac{H^2}{8\pi}=\frac{10^{10}}{8\pi}=4\times 10^8 \text{ ergs per c.cm.}$$

$$=490 \text{ ft.-lbs. per c. inch.}$$

117a. The Pull Between Two Magnetised Iron Surfaces.—In Fig. 144 are shown two iron cores between which a magnetic field is established. If B be the induction density [*assumed uniform*] in the gap, S sq. cms. the area of the gap, and x cms. the length of the gap, the magnetic energy stored in the gap will be—

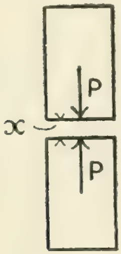

FIG. 144.

$$\frac{B^2}{8\pi} S\,x \text{ ergs.}$$

Let P be the pull in dynes due to this magnetic field and acting on each core face, and suppose the gap be increased by a small amount δx. The work done in moving the iron cores to increase the gap by this small amount is—

$$P\ \delta x \text{ ergs.}$$

The increase of energy of the magnetic field in the gap will be

$$\left[\frac{B^2}{8\pi} S(x+\delta x)-\frac{B^2}{8\pi} Sx\right] \text{ ergs}$$

$$=\frac{B^2 S\delta x}{8\pi} \text{ ergs.}$$

Hence $$P\ \delta x=\frac{B^2\ S\ \delta x}{8\pi};$$

or $$P=\frac{B^2\ S}{8\pi} \text{ dynes.}$$

This is a very important result, and enables, for example, the pull on a solenoid plunger to be calculated.

Expressed in lbs. weight—

$$P=\frac{2{\cdot}2B^2S}{8\pi\times 981\times 1000} \text{ lbs. weight,}$$

that is, $$P=0{\cdot}09\left(\frac{B}{1000}\right)^2 S \text{ lbs. weight,}$$

where S is in sq. cms. and B is lines per sq. cm.

In applying the above formula it is to be observed that the assumption was made that the value of B was constant throughout the gap. This assumption is approximately true if the length x of the gap is small compared with the area S of the gap. If x is not small the lines of force will spread out at the edge of the gap, and for an accurate calculation of P in this case the tubes of force should be mapped out and subdivided into small elements, so that B is constant over each element of volume of the gap and the energy of the tubes estimated for each such element.

118. Change in Direction of Magnetic Lines of Induction when passing from One Medium to Another.—In Fig. 145 is shown a tube of induction passing from a medium M_1 to a medium M_2, the boundary of the two media being indicated by AB.

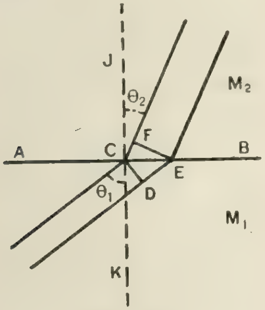

FIG. 145.

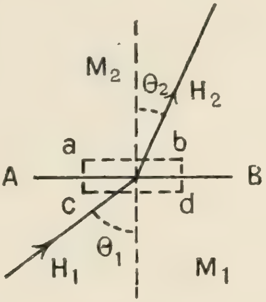

FIG. 146.

From what has been said in § 108 it follows that the flux in such a tube remains constant in passing across the boundary AB.

Let B_1 and μ_1 be respectively the induction density and permeability for medium M_1 and B_2 and μ_2, the corresponding quantities for medium M_2. Let θ_1 and θ_2 be the inclinations to the normal JK of the tube in the respective media. Draw the area CD perpendicular to the tube in M_1 and the area EF perpendicular to the tube in M_2.

Then $$CD = CE \cos \theta_1 : EF = CE \cos \theta_2.$$

Also $$\text{flux } (B_1 \times CD) = \text{flux } (B_2 \times EF).$$

Hence $$B_1 \times CE \cos \theta_1 = B_2 \times CE \cos \theta_2;$$

or $$B_1 \cos \theta_1 = B_2 \cos \theta_2.$$

A further condition which is to be fulfilled at the boundary surface is that the magnetic potential difference between two points very close to the boundary in one medium must be the same as the magnetic potential between two adjacent points close to the boundary in the other medium. Thus in Fig. 146 the p. d. between points a and b in medium M_1 must be the same as between the two points c and d in medium M_2 where a and c

are indefinitely close together, and d and c are indefinitely close together. To prove this consider the closed path $abcd$. If a unit charge be taken from a round this path so that it returns to a the work done during the transfer must be zero, since the final conditions of the unit charge are precisely the same as the initial conditions. If H_1 is the intensity of magnetic force in medium M_1, and H_2 the intensity in medium M_2, and if θ_1 and θ_2 are the respective inclinations to the normal JK of the direction of the forces in the two media (see Fig. 146), the work done in conveying unit charge from a to b is $H_2 \sin \theta_2 \, ab$, and the work done in conveying the charge from d to c is $-H_1 \sin \theta_1 \, dc$. The work done in moving the charge from b to d and from c to a is zero, since the distances bd and cd have been assumed to be indefinitely small.

Hence $$H_1 \sin \theta_1 ab = H_2 \sin \theta_2 cd = 0;$$

or $$H_1 \sin \theta_1 = H_2 \sin \theta_2.$$

Combining this condition with the first condition already arrived at—viz.:

$$B_1 \cos \theta_1 = B_2 \cos \theta_2$$

gives $$\frac{B_1}{H_1 \tan \theta_1} = \frac{B_2}{H_2 \tan \theta_2};$$

or $$\frac{\mu_1}{\tan \theta_1} = \frac{\mu_2}{\tan \theta_2}.$$

Suppose the medium M_1 is iron and the medium M_2 is air.

Then since $$\tan \theta_2 = \frac{\mu_2}{\mu_1} \tan \theta_1,$$

and $\frac{\mu_2}{\mu_1}$ is, in this case, a very small quantity, $\tan \theta_2$ will be practically zero—*i.e.*, $\theta_2 = 0$—or the lines will emerge from the iron into the air in a direction perpendicular to the surface of the iron. This result is very important, and it is necessary, when drawing lines of force, say, in the air gap between two iron surfaces, to so draw them that they are at right angles to the iron surfaces. In Fig. 147, for example, is shown a diagram of the lines of force drawn between a pole and the smooth armature of a dynamo. Compare also results of § 62.

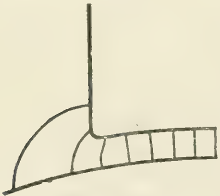

Fig. 147.

119. Effect of Temperature on Magnetism.—Temperature has a very great effect on the magnetic properties of iron, cobalt, and nickel. If a piece of iron is heated above about 1760° C. it ceases to be magnetic—that is, it cannot be magnetised and is not attracted by a magnet. As the iron cools down, however, it suddenly regains its magnetic properties at a temperature of about 760° C. The temperature at which the change takes place is known as the *critical temperature.* Steels of all kinds cease to be magnetic when heated above about 700° C. For high carbon steel the magnetic properties are regained when the steel cools below about 690° C., for low carbon steels the temperature is about 730° C., and for pure iron about 760° C., as stated above.

The sudden change in the magnetic properties of iron when passing through the critical temperature is also marked by other physical changes in the material. Thus, if a piece of iron is heated to a bright red heat and then allowed to cool it gradually assumes a dull red colour, and then as the temperature passes through the critical value the iron is seen to glow suddenly, showing that there has been a sudden release of heat by the iron at this temperature. This effect of sudden glowing at the critical temperature is termed "recalescence." In § 122 it is pointed out that the critical temperature is very important in connection with quenching and hardening steel for permanent magnets. Moreover, there is a sudden change in the electrical resistance of iron when it passes through the critical temperature (see Section III., § 143).

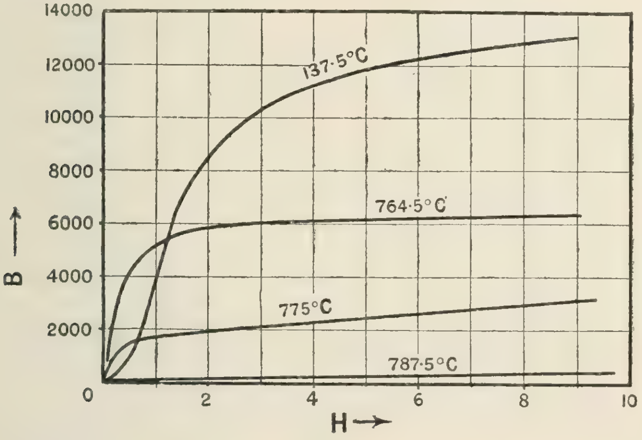

Fig. 148.

The critical temperature for cobalt is somewhat uncertain, but is higher than that for iron. For nickel the critical temperature is about 310° C.

In Fig. 148 are shown the magnetisation curves for iron for various values of the temperature. As the temperature is increased, the permeability μ diminishes until at a temperature of about 795° C. the value of $\mu=1$; *i.e.*, the iron becomes non-magnetic.

In Fig. 149 is shown the value of the permeability as a function of the temperature for various values of the magnetising force H.

It will be observed that for small magnetising forces the permeability *increases* with rise of temperature until the critical temperature is approached, when the rate of increase of permeability with temperature becomes first remarkably high, and then rapidly diminishes until at about 780° C. the permeability becomes about unity—that is to say, the iron becomes practically non-magnetic. It is noteworthy that for $H=0\cdot153$, the value of μ at 765° C. is about 12200, whereas an increase of temperature of only 15° C. reduces μ to unity.

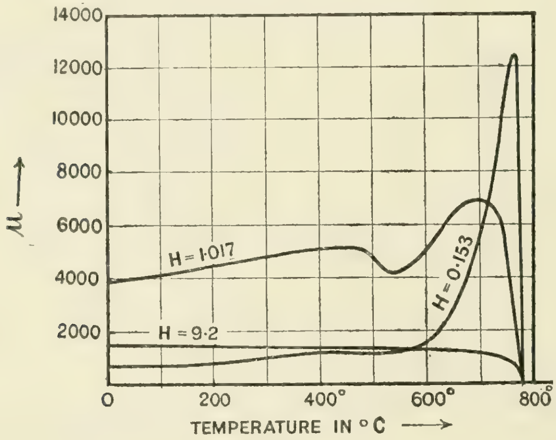

FIG. 149.

For stronger magnetising forces the permeability remains almost constant at its initial value until a temperature of about 100° C. lower than the critical temperature, when the permeability diminishes until at the critical temperature the magnetisation almost disappears. The results given in Figs. 148 and 149 are from a paper by Dr. D. K. Morris in the *Phil. Mag.*, 1897, vol. xliv.

120. Magnetic Properties of Cobalt and Nickel.—An average magnetisation curve for cobalt is shown in Fig. 150, and is similar to that of cast iron. The maximum intensity of magnetisation obtainable—*i.e.*, I_{max}—is about 1200, and the coercive force is about 12.

Fleming gives for the hysteresis loss for cast cobalt the expression $U_h=0\cdot01\ B_{max}^{1\cdot6}$ ergs per c.cm. per cycle.

An average magnetisation curve for nickel is also given in Fig. 150. The maximum intensity of magnetisation, I_{max}, for nickel is about 500.

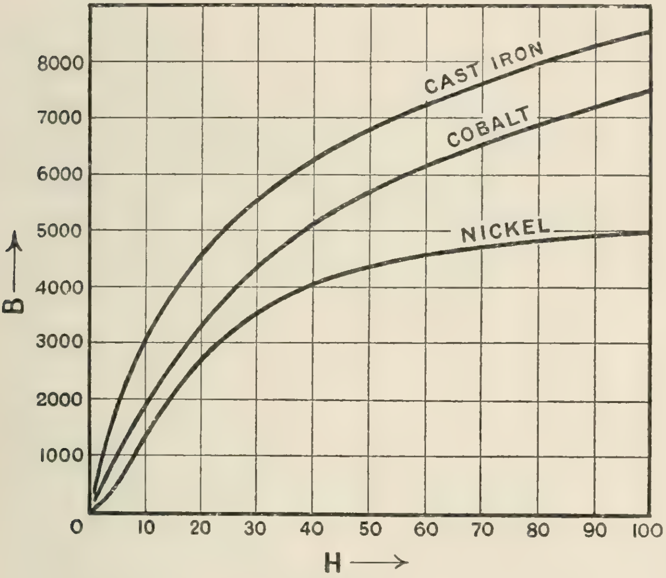

FIG. 150.

121. Paramagnetic and Diamagnetic Substances.—Faraday showed experimentally that most substances are acted on by a magnetic field, and bodies can be arranged in two distinct classes according to the nature of their action when placed in a magnetic field. In one class are substances such as iron, cobalt, and nickel, whose axis of magnetisation is in the same direction as the magnetising force, and in the other class are substances such as bismuth, whose magnetisation acts directly opposite to the magnetising force (see Fig. 151).

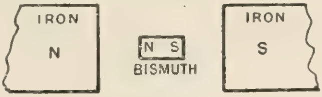

FIG. 151.

Substances which act like iron are termed *paramagnetic* or, more briefly, *magnetic*, and substances which act like bismuth are termed *diamagnetic*.

Although probably all substances are either magnetic or diamagnetic, it is to be observed that only iron, cobalt, and nickel are markedly susceptible to magnetism, all other substances being relatively extremely feebly influenced even by the most intense fields.

In § 108 the magnetic susceptibility was defined as—

$$\kappa=\frac{I}{H},$$

and the permeability as—

$$\mu=\frac{B}{H}=\frac{4\pi I+H}{H}.$$

from which $$\mu=1+4\pi\kappa.$$

For magnetic bodies κ is positive and for diamagnetic bodies κ is negative. Since κ is extremely small for diamagnetic bodies, the permeability is never negative, although it becomes slightly less than unity for diamagnetic substances.

The following table gives a list of a few prominent substances in each class:

Magnetic.	*Diamagnetic.*
Iron	Bismuth
Cobalt	Sulphur
Nickel	Water
Manganese	Alcohol
Platinum	Zinc
Oxygen	Hydrogen
Air	

Bismuth appears to possess the property of diamagnetism to a greater degree than any other substance, but even for bismuth the susceptibility κ is only about $-\frac{1}{80000}$.

122. Permanent Magnets.—Permanent magnets are those which retain a high degree of magnetisation when the magnetising force is removed.

The essential requirements for permanent magnets are that the remanent magnetism should be high, and also that the coercive force should be high.

For permanent magnets for modern requirements steel which shows lower values than $B_{rem}=10000$: $H_c=60$ is not suitable, these values being obtained from a hysteresis loop taken on a ring of the steel—that is to say, the conditions such that the demagnetising force is zero.

Tungsten steel is used by almost all makers of permanent magnets, and in Fig. 152 is shown a hysteresis loop for such a quality of steel, from which it will be seen that the value of B_{rem} is about 12000 lines per sq. cm., and the value of the coercive force H_c is about 65.

If steel is of such a composition that it can harden, it hardens if quenched at a temperature above recalescence. If it is cooled slowly through the recalescent stage, and then quenched, it is not thereby hardened, or only imperfectly hardened. Moreover, if it is quenched below the recalescent stage the resultant remanent magnetism and coercive force is less than if quenched above the recalescent stage.

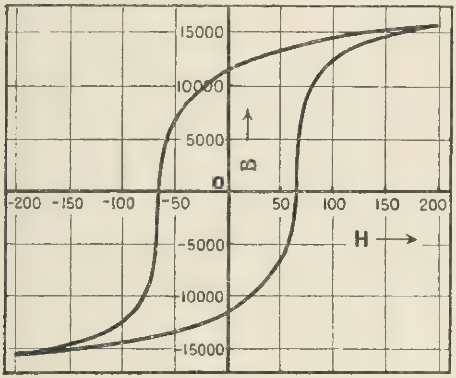

FIG. 152.

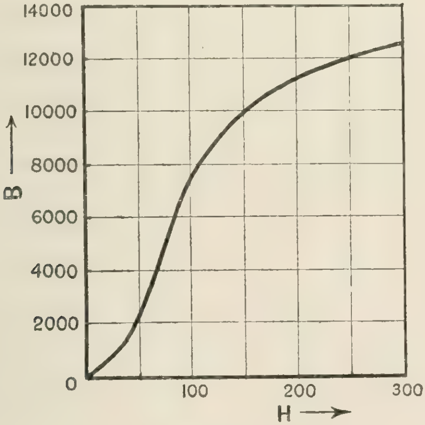

FIG. 153.

Thus, in the case of an annealed bar of steel of 20 diameters long, the value of I_{rem} was only 85, and the coercive force H_c only 8. This steel was then heated and quenched below the transformation temperature. The value of I_{rem} was then 130, and H_c was 14. When quenched above the transformation temperature I_{rem} became 410, and H_c 52.

These results show that in preparing steel for permanent magnets it is essential that the quenching should be performed at a temperature above that of recalescence.

In Fig. 153 is given a magnetisation curve of steel used by Messrs. Edgar Allen and Co. for permanent magnets.

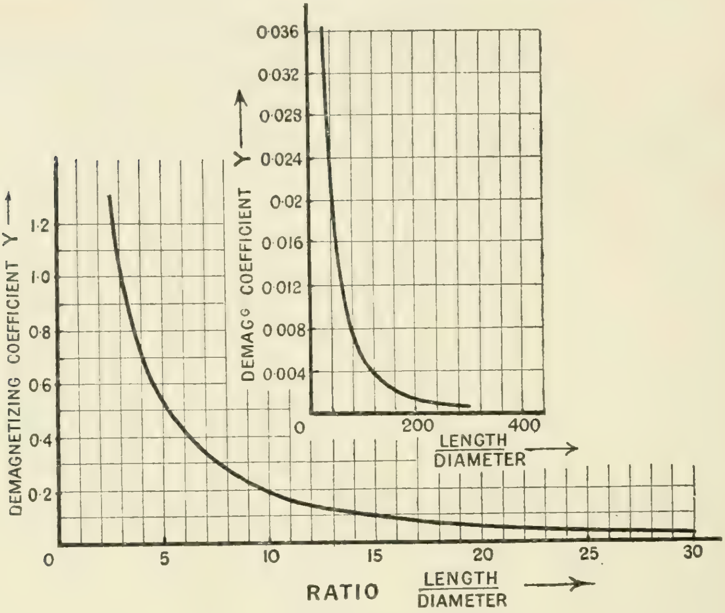

Fig. 154.

123. Self-Demagnetising Effect of Permanent Magnets.—

It was pointed out in § 110 that the self-demagnetising force at the middle of a long cylindrical magnet is small, whereas in the case of an iron sphere the self-demagnetising force may be very great. In the case of a cylindrical bar magnet of given ratio $\frac{\text{length}}{\text{diameter}}$ the self-demagnetising force at the middle of the bar may be estimated by comparison with the calculated results for ellipsoids, or it may be found directly by experiment.

If I is the intensity of magnetisation of the magnet, the demagnetising force may be written—

$$H_d = YI,$$

where Y is the demagnetising coefficient.

In Fig. 154 is shown the relationship between the ratio $\frac{\text{length}}{\text{diameter}}$ and the demagnetising force at the middle of the bar.

As further data on the influence of the ratio of $\frac{\text{length}}{\text{diameter}}$ as affecting the value of the remanent magnetism the following results given by Mme. Curie are of interest.

TUNGSTEN STEEL [5·5 PER CENT. TUNGSTEN, 0·59 PER CENT. C.] QUENCHED AT 770° C. COERCIVE FORCE, $H_C = 74$.

Ratio $\frac{\text{length}}{\text{diameter}}$	20	23·5	Infinity (*i.e.*, a closed ring).
B_{rem}	7000	8500	10700

CARBON STEEL [0·84 PER CENT. CARBON] QUENCHED AT 770° C. COERCIVE FORCE, $H_C = 53$.

Ratio $\frac{\text{length}}{\text{diameter}}$	20	22	71	Infinity (*i.e.*, a closed ring).
B_{rem}	5300	6000	7300	8100

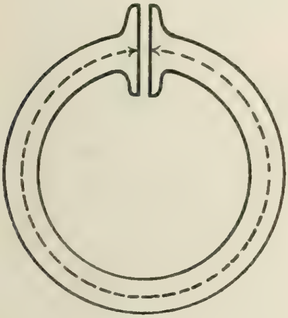

FIG. 155.

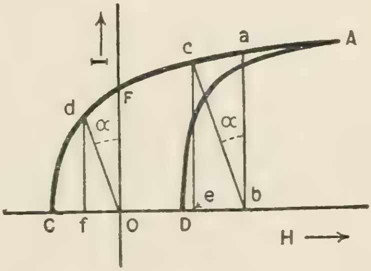

FIG. 156.

For split ring magnets (Fig. 155), such as are used in electricity meters, the demagnetising effect is nearly proportional to the width of the gap.

Let l_g be the length of the gap and l_m the length of the mean magnetic circuit in the iron.

Let A_g be the area of the gap, and A_m the cross-section area of the ring.

The demagnetising coefficient is—

$$\frac{1}{\lambda} 4\pi \left[\frac{l_g}{l_m} \frac{A_m}{A_g}\right],$$

where λ is the ratio of the flux at the middle of the magnet to the useful flux crossing the gap.

For magnets to be both powerful and constant the quantity in brackets should be not greater than about $\frac{1}{70}$.

The following graphical method for obtaining the approximate effect of self-demagnetisation of a magnet is given by Ascoli.

Let ABC (Fig. 156) be part of an I-H loop for a closed ring of steel—*i.e.*, for a specimen with zero self-demagnetising effect.

Now suppose a bar of the same steel is magnetised to the condition defined by the point A in Fig. 156, and let the value of H be then reduced to O*b*. The intensity I would, in the case of the closed ring, be given by *ab*. For the bar, however, there is a self-demagnetising effect, and if *bc* be drawn so that tan α=the demagnetising coefficient Y, *cc* gives the actual value of I for the bar when the magnetising force is *ob*. If H is reduced to zero the value of I_{rem} becomes *fd* for the bar, whereas for the closed ring specimen it would be OF.

For much valuable information on permanent magnets the reader is referred to a paper by Professor S. P. Thompson, *Journal of Inst. of Elec. Eng.*, 1913, vol. l. Much of the information in the foregoing paragraphs on permanent magnets has been derived and reproduced by permission from that paper.

Note on Cobalt Steel.—In *The Electrician*, vol. lxxxv., 1920, p. 706, an account is given of a new and remarkable magnet steel alloy known as " K.S. Magnet Steel," which was invented by H. Takagi and K. Honda. This alloy contains cobalt, tungsten, and chromium, a favourable composition being: carbon, 0·4 to 0·8 per cent.; cobalt, 30 to 40 per cent.; tungsten, 5 to 9 per cent.; chromium, 1·5 to 3 per cent.; the best quenching temperature for the steel being 950° C.

In one sample of this steel the value of I_{rem} =920, and the value of H_c=226. The coercive force is thus about three times as great as that of the best tungsten steel hitherto known, and the residual magnetism is also greater than that of tungsten steel.

CHAPTER IX

THE EARTH'S MAGNETIC FIELD—THE NAVIGATION COMPASS

124. The Earth's Magnetic Field.—As was mentioned in § 70, it is a property of a magnetised steel rod or needle freely suspended so as to move in a horizontal plane (*i.e.*, a compass needle), that it points approximately geographically north and south. This property has been known from a very early date (about 2000 B.C.) and was formerly thought to be due to the special attraction of the pole star for iron. It was pointed out, however, by Guilbert in the seventeenth century that this directive property of a compass needle is due to the earth itself being a magnet, in consequence of which the neighbourhood of the earth is a magnetic field.

The magnetic field due to the earth, although practically uniform throughout any such space as a building or rooms in which there is no magnetic substance, varies greatly in intensity and direction from one place to another on the earth's surface, and even in one place it changes from time to time, whilst "magnetic storms" occasionally occur, and indicate violent disturbances of the earth's field.

125. The Direction and Intensity of the Earth's Field.—In the British Isles the direction of the earth's field is roughly geographically north and south, but it makes a large angle with the horizon, the direction of the field pointing downwards.

The *magnetic meridian* is the vertical plane in which the direction of the earth's magnetic force lies.

The *astronomical meridian* is the vertical plane in which the geographical North and South Poles lie.

The direction of the earth's field is defined by two angles—viz., the *declination*, and the *inclination* or *dip*.

The *declination* at a place is the angle which the magnetic meridian makes with the geographical meridian at that place.

The *inclination* or *dip* is the angle which the direction of the earth's force makes with its projection on a horizontal plane. It is the dip of a needle, uniformly magnetised, in the direction of its length, and freely suspended at its centre of gravity.

The earth's magnetic field at a given place (and time) is defined by the following quantities:

(1) The intensity; (2) the declination; (3) the inclination or dip.

If the value of the horizontal component H of the earth's field is known, and also the dip δ, the intensity F is given by—

$$F = \frac{H}{\cos \delta}.$$

126. Measurement of the Horizontal Component H of the Earth's Magnetic Field.—The value of the horizontal component H may be found in several ways, but only one method will be described—viz., that due to Gauss.

This method consists in measuring (1) the angle through which a very small, freely suspended, horizontal magnetic needle is deflected by a magnet in a given position, and (2) the period of vibration of the same magnet as used in (1), when suspended horizontally in the earth's field, and free to oscillate about a vertical axis.

The measurement of (1) gives an equation involving the ratio of the magnetic moment M of the magnet and the horizontal component H of the earth's field.

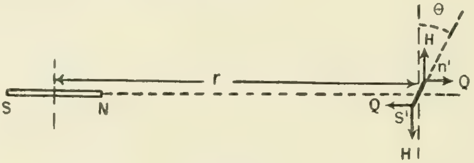

FIG. 157.

The measurement of (2) gives an equation involving the product of the same two quantities. Hence the value of H may be deduced from these two equations, and incidentally the same two equations give the value of the magnetic moment M of the magnet.

Experiment 1. *Determination of the Ratio* $\frac{M}{H}$.—In Fig. 157 let NS be the deflecting magnet of moment M. If this magnet is of a length $2l$, which is great compared with its lateral dimensions, say a cylindrical magnet of which the ratio $\frac{\text{length}}{\text{diameter}}$ is about 100, it is very approximately correct to assume that $2l$ is the distance between the poles, and that the strength of the poles is $\frac{M}{2l}$ (see § 91).

Let $n^1 s^1$ be the small, freely suspended magnetic needle, the centre of $n^1 s^1$ being on the prolonged axis of NS, and let M^1 be the magnetic moment of the needle, and $2l^1$ its length. If the ratio $\frac{\text{length}}{\text{diameter}}$ of the needle be sufficiently large, the strength of each pole may be taken as $\frac{M^1}{2l^1}$.

Let θ be the angle which the needle makes with the magnetic meridian when the relative positions of needle and magnet are as shown in Fig. 157.

The deflecting force on the pole n^1 will be the resultant of the repulsion due to N, and the attraction due to S. If the distance r between the centres of magnet and needle be great compared with l it will be very approximately correct to assume this resultant magnetic intensity Q at n^1s^1 due to NS to be parallel to the line through SN, as shown in Fig. 157. The magnitude of the force on n^1 is given by the expression—

$$\frac{M^1}{2l^1}Q=\frac{MM^1}{4ll^1}\left[\frac{1}{(r-l)^2}-\frac{1}{(r+l)^2}\right]=M\frac{M^1}{l^1}\frac{r}{(r^2-l^2)^2},$$

that is

$$Q=M\frac{2r}{(r^2-l^2)^2}.$$

This will also be the resultant force on the pole s^1 of the attraction due to N, and the repulsion due to S. Hence the effect of the magnet NS on the needle n^1s^1 is a deflecting couple of moment—

$$\frac{M^1}{2l^1}Q2l^1\cos\theta.$$

This couple will be balanced by a couple due to the horizontal component H of the earth's field, which tends to set the needle in the magnetic meridian. The magnitude of this couple will be—

$$\frac{M^1}{2l^1}H2l^1\sin\theta=M^1H\sin\theta.$$

For equilibrium of the needle these two couples must be equal.

Hence

$$\frac{M^1}{2l^1}Q2l^1\cos\theta=M^1H\sin\theta;$$

that is

$$\tan\theta=\frac{QM^1}{M^1H}=\frac{Q}{H}$$

or

$$\tan\theta=\frac{M}{H}\frac{2r}{(r^2-l^2)^2};$$

and

$$\frac{M}{H}=\frac{(r^2-l^2)^2}{2r}\tan\theta.$$

If r is very large compared with l this expression reduces to—

$$\frac{M}{H}=\frac{r^3}{2}\tan\theta.$$

It is advisable to take another reading for θ by turning the magnet NS end for end, and the mean of the two readings is then taken as the true value of the deflection.

The values of θ may be measured by the movement of a spot of light reflected on to a scale from a mirror attached to the needle n^1s^1.

Experiment 2. *Determination of the Product MH.*—For this purpose the magnet which was used in Experiment 1 for the purpose of deflecting the small suspended needle is now hung in a horizontal position by means of a single fibre of unspun silk, precautions being taken that there is no torsion on the fibre when the magnet is at rest in its position of equilibrium. The magnet is then deflected by approaching a small magnet, care being taken that a true oscillation is obtained, and not combined with a pendulum motion. The time of a complete oscillation is taken when the amplitude of vibration has died down to a small amount, say about 3° deflection each side of the neutral position.

If θ represents the angular deflection of the oscillating magnet from the neutral position, then for small values of θ and neglecting the damping effect of the air, etc., the equation of motion is—

$$\frac{d^2\theta}{dt^2} = -\frac{\mathrm{MH}}{\mathrm{K}}\theta,$$

where K is the moment of inertia of the magnet round an axis through its centre and at right angles to its length.

The solution of this equation is—

$$\theta = \mathrm{A}\sin\left[\sqrt{\frac{\mathrm{MH}}{\mathrm{K}}}\,t - \alpha\right],$$

where A and α are constants.

The time of a complete oscillation is therefore—

$$\mathrm{T} = 2\pi\sqrt{\frac{\mathrm{K}}{\mathrm{MH}}}.$$

Since, as was assumed in Experiment 1, the diameter of the magnet is small compared with its length the moment of inertia may be written—

$$\mathrm{K} = \frac{\mathrm{W}l^2}{3},$$

where W is the mass of the needle.

Hence $$\mathrm{MH} = \frac{4\pi^2 l^2 \mathrm{W}}{3\mathrm{T}^2}.$$

Combining this equation for the product MH with the equation given by Experiment 1 for the quotient $\frac{\mathrm{M}}{\mathrm{H}}$ gives—

$$\mathrm{H}^2 = \frac{8}{3}\,\frac{\pi^2 l^2 r \mathrm{W}}{\mathrm{T}^2(r^2 - l^2)^2 \tan\theta};$$

or, neglecting l compared with r—

$$\mathrm{H}^2 = \frac{8}{3}\,\frac{\pi^2 l^2 \mathrm{W}}{r^3 \mathrm{T}^2 \tan\theta}.$$

Similarly $$M^2=\frac{2}{3}\,\frac{\pi^2 r^3 l^2 W \tan\theta}{T^2}.$$

The data of the earth's field in England are—

$$H=0\cdot18:\ \text{dip}\ \delta=67^\circ\ 14':\ \text{intensity}\ F=0\cdot475.$$

For a more detailed account of this and other standard measurements the reader is referred to Professor A. Gray's book on "Absolute Measurements in Electricity and Magnetism."

For the practical importance of a knowledge of H, see § 213, Section IV.

The total intensity F of the earth's field is connected with the dip δ and the horizontal component H by the expression—

$$H=F\cos\delta.$$

127. The Navigation Compass.—Essentially the compass consists of a magnetic needle or combination of needles balanced so as to swing horizontally on a pivot. Secured to the needle and swinging with it is a circular card marked with the thirty-two points, the point marked N being over the northern end of the magnetic axis of the needle. The pivot on which the compass card is supported is secured to the bottom of the bowl of the compass, and a glass covering to the bowl protects the compass from the weather.

If a single magnetic needle were employed for the compass, the oscillations of the card due to the movement of the ship would be so violent as to render the instrument impracticable. To reduce this effect, the method was early adopted of using several needles arranged so that the moment of inertia of the system about all diameters is approximately the same. Fig. 158 shows the arrangement of four needles adopted by the Admiralty in an early form of standard compass.

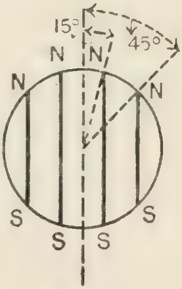

Fig. 158.

Incidentally, the use of a composite needle is favourable to a more complete correction for the disturbing effect of the ship's magnetism, as referred to below.

On the introduction of iron ships, it was found that the ship itself became magnetised by the earth's field, and produced serious deflections of the compass needle. The influence of the ship's magnetism on the compass may be very effectively eliminated by corrector magnets, and by masses suitably placed in the neighbourhood of the compass (see § 128).

128. The Deviation Effects on the Compass due to the Ship's Magnetism and the Means of Eliminating this Effect.—The effect of the ship's magnetism may be considered under three heads —viz.:

(*a*) The horizontal component of the magnetism of the soft iron in the ship.
(*b*) The vertical component of the magnetism of the soft iron in the ship.
(*c*) The permanent magnetism of the ship.

(*a*) *The Horizontal Component of the Magnetism of the Soft Iron in the Ship.*—Suppose in Fig. 159 the magnetism due to this effect is represented by a horizontal magnet NS level with the compass card. As the ship swings round the effect will be equivalent to moving the magnet round the compass needle, always keeping its axis parallel to the magnetic meridian.

In Fig. 159 eight such positions are shown, and numbered 1 to 8. It will be obvious that the positions 2 and 6 produce an easterly deflection of the compass, whilst the positions 4 and 8 produce a westerly deflection. The positions 1, 3, 5, and 7 do not deflect the compass.

The deflection therefore changes from quadrant to quadrant, and the error due to this cause is spoken of as the *quadrantal error.* It is corrected by arranging cast-iron spheres one on each side of the compass, the exact position for correct adjustment being found by trial. The spheres are about 8 inches diameter, placed on either side of the compass at about 12 inches from the card centre, and level with the card. These spheres become magnetised by the earth's field, and their magnetic axis is always parallel to the magnetic meridian.

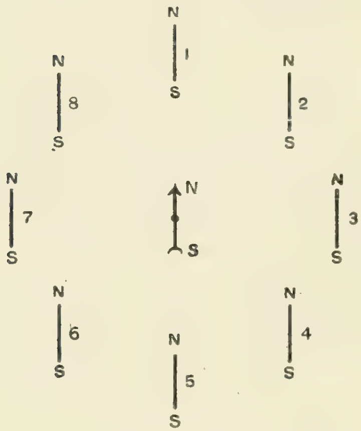

Fig. 159.

The field due to these spheres on the region swept out by the compass needles can only be taken as uniform if the needles are very short, and this is a very important consideration in fixing the size of the needle. If the needles are not kept short the correction introduced by the cast-iron spheres will not be true for all directions of the ship's head.

It is to be noted in this connection that it has been proved both mathematically and experimentally that the use of a composite compass needle, such as is shown in Fig. 158, greatly reduces the error due to the non-uniformity of the connecting field.

(*b*) *The Vertical Component of the Magnetism of the Soft Iron in the Ship.*—The deviating effect on the compass of the vertical component of the magnetism of the soft iron in the ship may be considered equivalent to a vertical magnet, of which the upper pole is S in the northern hemisphere and arranged either ahead or abaft the compass.

Suppose in Fig. 160 the effect of the vertical component is equivalent to a vertical magnet ahead of the compass, the upper pole being S if the ship is in the northern hemisphere. As the ship's head turns round the effect of this on the deviation of the compass is the same as if the rod were carried round the compass.

Eight such relative positions are shown in Fig. 160.

It is clear from this figure that for positions 2, 3, and 4 the deviation of the compass will be easterly, and for positions 6, 7, and 8 the deviation will be westerly—that is, the direction of the deviation changes for each semicircle of change of the ship's head from north to south, and is called the *semicircular deviation*, since the direction of the deviation remains the same for a semicircle of swinging of the ship.

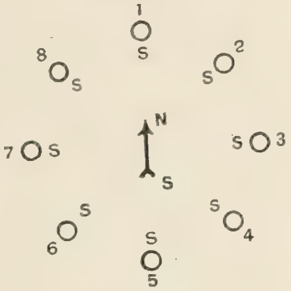

FIG. 160.

The correction for this deviation is provided by a vertical rod of soft iron fixed to the binnacle of the compass, the upper end of the rod being about level with the compass card. This rod is about 3 inches diameter, and perhaps 16 inches long, and is known as Flinder's Bar. The correct position of the rod for elimination of the semicircular deviation is best found by trial.

(*c*) *The Permanent Magnetism of the Ship.*—During the building of an iron ship the hammering and riveting of the plates converts them into permanent magnets, and the whole ship, when built, acts like a permanent magnet, with its magnetic axis in a direction depending on the direction in which the ship's head lay when building. A considerable degree of this magnetism is lost during a long voyage, probably on account of the buffeting of the ship by the sea, and the magnetism which finally remains is the *permanent* magnetism. The magnetism lost during the trials and first voyage is called the *sub-permanent* magnetism.

The correction for the deviation of the compass due to this permanent magnetism is obtained by fixing suitable permanent magnets in the neighbourhood of the compass.

129. Lord Kelvin's Compass.—This compass was originally patented in 1877, and is now almost universally used. The principal characteristic features are:

(*a*) A compass card of very small mass and long vibration period, by which the excessive oscillation of the card due to the movements of the ship is reduced to a minimum. The period of vibration of the card in England is about 40 seconds.

(*b*) Very short, light needles used, so that the correction for the ship's magnetism may hold for all courses of the ship.

(*c*) Needles of small magnetic moment used, so that there may be no appreciable induction effect of the needle on the correcting masses of iron.

The arrangement of the compass card and needles is shown in Fig. 161. There are eight needles, of which the longest is $3\frac{1}{4}$ inches long. The central part of the card is cut away, and only the rim, strengthened by a rim of aluminium, is used. The greater proportion of the weight of the card is thus in the rim, and this gives a long period of oscillation of the card. A pivot of iridium and cap of sapphire is used, and the card is secured to the cap by means of thirty-two radial silk threads.

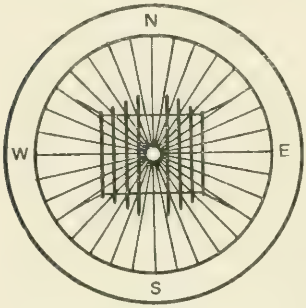

Fig. 161.

The eight magnetic needles are supported from some of these radial threads as shown in Fig. 161.

The centre of gravity of the complete card is well below the point of suspension.

The compass bowl is of brass, and is supported by gimbals resting on knife edges. It is weighted underneath by an expansion chamber filled with castor oil to keep it steady.

130. Sperry's Gyroscopic Compass.—This type of navigation compass has found extensive application in recent years. The principle of action depends upon the use of a gyroscope, and the compass indicates true geographical north at all times. It is entirely non-magnetic, and is, therefore, not subject to any errors caused by magnetic variation. Further, it is not susceptible to magnetic surroundings, and the directive force for this type of compass is claimed to be about three hundred times greater than that of a magnetic compass. The gyroscope compass is specially suitable for submarines since the large masses of iron used in the equipment of the vessel, many of which are necessarily close to the compass, would make the use of a magnetic compass impracticable.

For a detailed account of the action of the Sperry compass reference should be made to A. E. Gott, *Nautical Magazine*, September, 1916.

SECTION III

DIRECT CURRENT ELECTRICITY

CHAPTER X

FIRST PRINCIPLES

131. Electric Current.—In Fig. 162 are shown the tubes of strain due to a charged condenser, consisting of two parallel plates A and B separated by insulating material (*e.g.*, air). It is assumed in connection with what follows that the plate A is charged with positive electricity, and the plate B with an equal quantity of negative electricity. Each tube of strain is in equilibrium under the tension along the tube, and the pressure of neighbouring tubes at right angles to its length, as was explained in § 61, Section I.

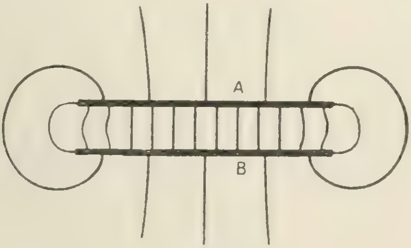

FIG. 162.

Suppose that one end of a wire be joined to the plate A and one end of another wire to the plate B, the free ends *cd* of the wires being kept apart, as shown in Fig. 163. The distribution of the tubes of strain will change from that shown in Fig. 162 in order to conform with the requirement that the wire connected to each plate must assume the potential of that plate, and the tubes of strain will then be somewhat as shown in Fig. 163.

Now let the free ends *cd* of the wires be connected together. The tension along the tubes whose ends are on the wires will cause the equal and opposite electric charges on which each tube ends to slide along the wires until they coalesce and neutralise each other. At the same time, the tension along each such tube causes it to shrink as the ends approach each other, and the tube thus disappears. As each tube disappears the equilibrium of the remaining tubes is thereby disturbed, and the ends of the tubes move from the condenser plates to the respective wires, approach and coalesce, the corresponding tubes shrinking and eventually disappearing, as explained.

In this way all the tubes disappear, and the condenser becomes discharged. It is thus seen that the positively charged ends of the tube move from the plate A along the wire towards the plate B, and the negatively charged ends move from the plate B along the wire towards the plate A.

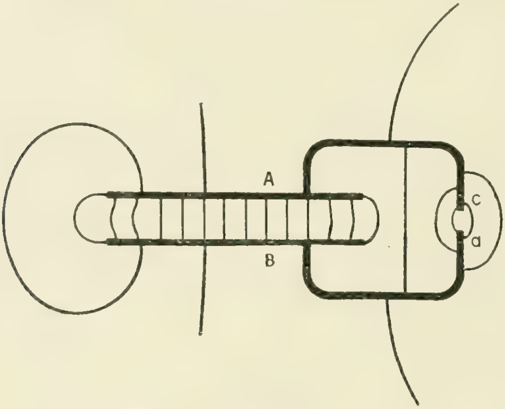

Fig. 163.

The movement of the electric charges along the wire constitutes an **electric current** in the wire, and as long as the current is maintained, the wire itself and its immediate neighbourhood exhibit special properties to which the current gives rise. For instance, the wire becomes heated and, as will be seen later (§ 145), the mechanical equivalent of the total heat developed during the discharge of the condenser is equal to the energy stored in the dielectric of the condenser before the wires were connected.

Suppose before the wires *cd* are joined the potential difference of the condenser plate is V volts, and the charge of the condenser Q coulombs. Then from § 36 it follows that

$$Q=VC,$$

where C is the capacity of the condenser in farads.

When the wires are joined the work done by the electric forces in the passage of the small quantity of electricity δQ along the wire is (§ 28)—

$$V\delta Q \text{ ergs};$$

and the work done in discharging the condensers by an amount $Q - Q_1$ will be—

$$= -\int_{Q}^{Q_1} V\,dQ = -\int_{Q}^{Q_1} \frac{Q}{C}\,dQ = \frac{1}{2}\,\frac{Q^2 - Q_1^2}{C} \text{ ergs,}$$

the minus sign in the integration being taken because the quantity of the charge is diminishing.

The expression just deduced also gives the amount by which the energy stored in the dielectric is diminished (§ 50).

This loss of energy by the electric field is accounted for by the development of an equivalent quantity of heat in the wires connecting the plates of the condenser, and also a relatively small amount of heat produced in the plates of the condenser due to the movement of the charged ends of the tubes over the plates.

A further consequence of the flow of the electric current is that a magnetised needle brought in the neighbourhood of the current experiences a magnetic force due to the current, as will be more fully explained in Section IV.

This magnetic force is different according to whether the plate A or the plate B is the positively charged plate. It is conventionally agreed to consider the direction of the current as being from the positively charged plate to the negatively charged plate, although, as has been explained, the current consists really in a movement of positive and negative electricity in opposite directions—*i.e.*, towards each other.

132. Strength of the Current.—If at any cross-section X of the wire joining the plates AB (Fig. 163), δQ units of positive electricity pass in the direction from A towards B, or δQ units of negative electricity pass in the direction from B towards A during the time δt, the strength of the current at X is—

$$\frac{\delta Q}{\delta t}.$$

If δQ is measured in coulombs and δt in seconds, the current is measured in *ampères*.

> **Hence a current of one ampère flows across any section of a wire when electricity passes that section at the rate of one coulomb per second.**

If the current varies from moment to moment, as is very frequently the case in practice, the strength of the current at any time is given, in the notation of the differential calculus, by the expression—

$$\frac{dQ}{dt} \text{ ampères.}$$

133. Production of a Current by means of Electric Cells.—

Volta showed that by the mere contact of two dissimilar metals the metals become oppositely charged, one becoming positively electrified and the other negatively.

Volta assumed that the contact surfaces of the metals become the seat of a force which he termed the **electromotive force** (usually written e. m. f.), which tends to transfer positive electricity across the surface in one direction, and negative electricity across the surface in the opposite direction, the magnitude of this contact e. m. f. depending only on the nature of the metals in contact.

As an example of this phenomenon, suppose a sensitive electroscope is provided with a copper disc-shaped cap, and that a disc of zinc is provided, fixed to an insulating handle, and laid on the copper disc (see Fig. 164). If the zinc disc is then removed by means of its handle the leaves of the electroscope will diverge, the electroscope having become *negatively* charged

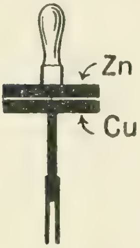

Fig. 164.

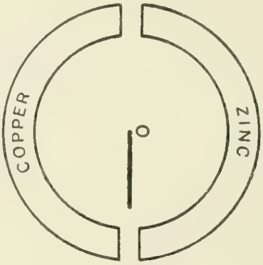

Fig. 165.

Another experiment was devised by Lord Kelvin to demonstrate the development of a potential difference between two dissimilar metals in contact. Two semicircular rings of copper and zinc were obtained so that when placed in contact they formed a complete circle. A light needle was suspended, as shown in Fig. 165, so that it was movable about a point in the axis of the metal ring and the needle was charged by means of a Leyden jar. If the half-rings were uncharged and separate, as shown in Fig. 165, no movement of the needle took place.

When, however, the two half-rings were brought into contact the charged needle moved so that if the charge on the needle was negative it turned towards the zinc half-ring. When the needle was positively charged it turned towards the copper half-ring.

The explanation of this development of electricity on metals in contact has provoked much discussion, and so far no conclusively satisfactory test experiment has been carried out to definitely decide what gives rise to the electric charges on dissimilar metals which are in contact. It is, however, generally considered that chemical action is the immediate cause in some such way as follows:

The oxygen of the atmosphere consists of electrically neutral molecules, and each molecule consists of one positively charged atom and one negatively charged atom. The zinc in contact with the atmosphere becomes oxidised, and results in the splitting up of the electrically neutral molecules of oxygen, so that the negatively charged atoms combine with the zinc, and deliver their charges to the zinc, whilst the positively charged atoms surround the zinc, and thus an electric field is established round the zinc from the adjacent positively charged oxygen atoms to the negative electricity on the zinc. This action is assumed to go on until the electric forces thus developed balance the chemical forces tending to the oxidation of the zinc. Equilibrium is then established, and the oxidation ceases. This condition of equilibrium is reached when the potential difference between the zinc and the immediately adjacent air is about 1·8 volts. A piece of copper is subject to a similar oxidising action of the atmosphere, but in this case equilibrium of the chemical and electric forces is established when the potential difference between the copper and the immediately adjacent air is about 0·8 volt. This is in accordance with the fact that zinc is known to be chemically more active than copper (see § 157).

When the zinc and copper are placed in contact, it is assumed that the potential of the two metals is equalised in accordance with the general property of conductors, and the potential of the zinc rises, and that of the copper falls, negative electricity passing across the contact surface from zinc to copper. The electric forces between the zinc and surrounding air then no longer balance the chemical tendency to oxidise, and the zinc oxidises further, and at the same time it is possible that the copper deoxidises until the conditions are again established that the potential difference between the zinc and neighbouring air is 1·8 volts, and the potential difference between the copper and neighbouring air is 0·8 volt. The potential difference between the air in the neighbourhood of the zinc and that in the neighbourhood of the copper is thus 1 volt, and a field is established of which the positive boundary is the air in the neighbourhood of the zinc. In testing for the nature of the charge on the zinc disc in the experiment illustrated in Fig. 164 the explanation just given would account for the zinc being apparently charged with positive electricity.

Volta found that if a series of discs of various metals be placed in contact, just the same e. m. f. of contact is developed between the first and last metal of the series as if these two were in direct contact—that is to say, there is no accumulation of potential by such an arrangement. He also found that all metals may be arranged in an e. m. f. series, such that each metal becomes positive when placed in contact with the one immediately below it. Such a series is as follows:

	Volt.
Zinc	—
Lead	0·210
Tin	0·071
Iron	0·319
Copper	0·150
Platinum	0·230
Carbon	0·115

The number opposite the name of each metal represents the contact e. m. f. in volts which will be developed between that metal and the one immediately above it in the series when the two are placed in contact. Thus, if lead and zinc are placed in contact there is a contact e. m. f. of 0·21 volt developed between the metals, the zinc being positive to the lead.

Further if, say, a piece of tin and a piece of zinc are in contact, the contact e. m. f. developed will be given by the e. m. f. of zinc to lead + the e. m. f. of lead to tin—*i.e.*,

$$0{\cdot}210 + 0{\cdot}071 = 0{\cdot}281 \text{ volt}$$

Hence, if a closed circuit be made of different metals, as shown in Fig. 166, the e. m. f. between zinc and lead is 0·210 volt.

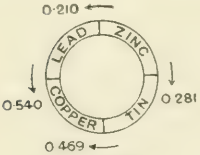

FIG. 166.

The e. m. f. between zinc and tin is—

$$0{\cdot}210 + 0{\cdot}071 = 0{\cdot}281 \text{ volt.}$$

The e. m. f. between tin and copper is—

$$0{\cdot}319 + 0{\cdot}150 = 0{\cdot}469 \text{ volt.}$$

The e. m. f. between lead and copper is—

$$0{\cdot}071 + 0{\cdot}319 + 0{\cdot}150 = 0{\cdot}540 \text{ volt.}$$

Hence the total e. m. f. round the closed circuit is zero, and this will always be the case in a closed circuit consisting of metals only, provided the temperature of each junction of two dissimilar metals is the same (see § 176).

One method of measuring the magnitude of the contact e. m. f. is by making the two pairs of quadrants of an electrometer (§§ 67 and 68) of the metals to be examined. On connecting the two quadrant pairs by a wire a deflection of the needle is obtained. By applying an external e. m. f. in the opposite direction through the connecting wire the deflection may be reduced to zero, and the applied external e. m. f. measures the contact e. m. f. of the dissimilar metals of the quadrants.

If, however, pairs of metallic discs, say of zinc and copper, are alternated with discs of cloth wetted with ordinary tap water (*i.e.*, not pure water) or dilute acid (Fig. 167), the effect of the p. d. of one pair of copper and zinc discs *is added to that of the next pair*, and thus by a sufficiently long series of pairs any desired p. d. may be obtained. Such a series forms an **electric battery**.

134. Condensing Electroscope.—Suppose an electroscope has a copper disc cap, of which the surface is coated with shellac, and suppose a second similar copper disc is provided, of which the face is also coated with shellac, and which is supported by an insulating handle. When the second disc is placed on the cap of the electroscope (see Fig. 168) the two discs together form a condenser of relatively large capacity, since the distance between the two discs is small (see § 40), and the arrangement is called a *condensing electroscope.*

Now if a disc of zinc and a disc of copper, separated by a layer of wet cloth, be taken, and a wire joined to each disc be allowed to touch respectively the upper and lower plates of the condensing electroscope, each plate of the electroscope assumes the potential of the disc of the voltaic element to which it is connected, and consequently a charge of electricity accumulates on each plate of the condensing electroscope. Now disconnect the wires, and raise the upper disc of the electroscope some distance. The capacity of the disc forming the cap of the electroscope is now enormously diminished, the potential rises accordingly, and the leaves of the electroscope diverge. If the nature of the charge be tested it will be found to be *positive* electricity. Similarly, the charge on the upper plate of the electroscope will be *negative* electricity.

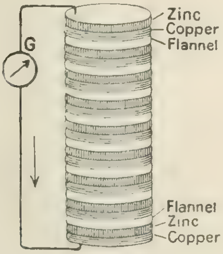

Fig. 167.

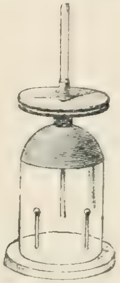

Fig. 168.

The object of coating each disc with an insulating layer of varnish is to avoid the "soaking in" effect referred to in § 49.

135. Voltaic Cell.—If a plate of zinc and a plate of copper be dipped side by side, but not touching, in a vessel of acidulated water, it is found that the plates differ in potential.

If a piece of copper wire be joined to each plate, and if these wires be connected to the alternate quadrant pairs of an electrometer, a deflection corresponding to about 1 volt is indicated, the potential of the wire from the copper plate being the higher. The explanation of the development of this potential difference is similar to that given in § 133, it being observed in the present case that the oxidisation of the parts of the plates in contact with the liquid is produced by the liquid, and, further, that the liquid is a conductor of electricity. Similar results are observed if almost any two dissimilar metals are substituted for the copper and zinc, and if instead of acidulated water ordinary tap water is used, or water containing a little salt in solution (and thus rendered conducting), the only difference being the magnitude of the observed potential difference of the plates. Such an arrangement of metals and conducting liquid is called an **electric cell** or **voltaic cell.**

The following lists refer to two different liquids, and each list is so arranged that if any two metals from the list be dipped in the liquid to which the list refers, the metal standing higher in the list is positive to the other:

Caustic Potash.	*Sulphuric Acid.*
Silver	Graphite
Nickel	Platinum
Copper	Gold
Iron	Silver
Bismuth	Copper
Lead	Antimony
Antimony	Bismuth
Tin	Nickel
Zinc	Iron
	Lead
	Tin
	Zinc

The p. d. between two plates of dissimilar metal dipped in a liquid conductor depends only on the nature of the metals, the liquid employed, and, to some extent, on the temperature. The p. d., however, does not depend on the absolute potential of the cell, and consequently, if a number of cells, say of copper and zinc, dipped in acidulated water, be connected " in series "—*i.e.*, the copper of one cell to the zinc of the next, and so on—the p. d. between the zinc of the first cell and the copper of the last is the sum of the individual p. d.'s of each cell. Such an arrangement of cells is called, as already stated in § 133, an **electric battery.**

If the plates of an electric cell be joined by a wire, an electric current will flow along the wire, as explained in § 131, the direction of the current being from copper to zinc along the wire, and from zinc to copper through the liquid of the cell. The passage of the current through the liquid of the cell results in the zinc becoming dissolved, and bubbles of hydrogen appear on the copper plate. The hydrogen bubbles forming on the copper plate give rise to an opposing e. m. f., which results in the diminution of the effective e. m. f. of the plates of the cell, and the current gets weaker. If the outside connecting wire joining the two plates be now broken, the hydrogen slowly disappears, and eventually the plates of the cell recover their normal p. d.

The formation of the bubbles of hydrogen on the copper plate is termed the **polarisation** (see § 159) of the copper plate, and is a serious disadvantage of such a simple cell. Numerous cells have been devised in which the polarisation effect is avoided (see § 137).

If ordinary commercial zinc be used in the cell it gradually dissolves, even when the outside circuit is not closed. The cause of this is the voltaic action between the zinc and the impurities, local electric cells being thus formed, and this effect is known as the "local action" of the cell. By smearing a little mercury over the surface of the zinc—*i.e.*, by *amalgamating* the zinc—this local action is prevented.

136. Seat of the Electromotive Force in a Voltaic Cell.—It is instructive to enquire a little more closely as to how the p. d. of the plates of a voltaic cell is produced, and what part the accompanying chemical action plays in the development of this p. d. Such an enquiry is also important in understanding electrolysis, which is the reverse action of that of a voltaic cell (see Chapter XII.). The following is a possible explanation of the phenomena, although the process can hardly be considered as completely understood in all its stages.

A molecule of water consists of two atoms of hydrogen, combined with one atom of oxygen, the atom of oxygen being charged negatively, and the hydrogen atoms charged positively, so that the molecule of water is electrically neutral. When zinc is dipped in water the chemical affinity of the zinc for the oxygen causes the two to unite, and the oxygen thus delivers up its charge to the zinc, and consequently the water and zinc become respectively charged with positive and negative electricity. As this action proceeds, the surface of the zinc becomes faced with a layer of positively charged hydrogen molecules, the condition of the zinc and the charged layer of hydrogen being probably similar to that between the plates of a charged condenser.

The action proceeds until the potential of the water is about $+1{\cdot}8$ volts with respect to the zinc, in which case the electric force developed between the water and the zinc balances the chemical affinity of the zinc for the oxygen, and a condition of equilibrium is thus reached, just as in the case of zinc in contact with air, as explained in § 133. The zinc oxide formed on the zinc is dissolved by the sulphuric acid in the water.

If, now, a copper plate be placed in the same vessel of water, a similar action takes place at the surface of the copper, but since copper is chemically less active than zinc, the copper attains a potential of about −0·8 volt with respect to the water, and since, as has been explained, the water has a potential of +1·8 volts with respect to the zinc, the resultant potential of the copper is 1·0 volt higher than that of the zinc, and *this is the e. m. f. of the cell.*

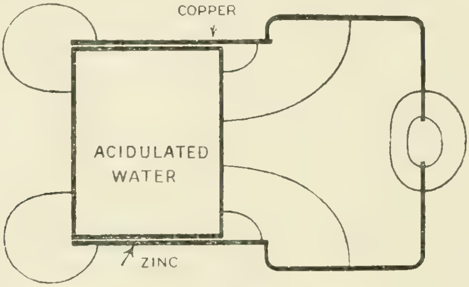

Fig. 169.

It is to be borne in mind that the acidulated water is a conductor, and it is in this respect that the action here considered differs from that considered in § 133.

In Fig. 169 some of the lines of force are shown sketched in for this condition. A wire is shown attached to each plate, and, of course, the wire assumes the same potential as the plate to which it is connected.

Now, suppose the two wires to be connected together. A current of electricity will flow along the wire from the copper plate to the zinc—*i.e.*, the copper plate will lose positive electricity, and the zinc plate will lose negative electricity (see § 131). The potential of the copper plate thus tends to be lowered, and that of the zinc plate to be raised. The chemical and electrical equilibrium at the surfaces of the zinc and copper is thus disturbed, and chemical action proceeds, so that more zinc combines with the oxygen of the water, thus releasing more positively charged hydrogen molecules. Possibly the copper oxide on the copper plate decomposes to some extent, the released oxygen recombining with the free hydrogen adjacent to the copper plate. The excess of chemically free hydrogen near the zinc then combines with the oxygen of the adjacent molecules of water, the combination being facilitated by the electric field existing in the water from the copper to the zinc plate. The hydrogen thus released from the molecule of water then combines with the oxygen of the next molecule, and so on, until eventually the hydrogen released from the molecules of water adjacent to the copper plate appear as bubbles of gas. In the body of the liquid, however, in between the zinc and copper plates, no bubbles of gas appear.*

137. Constant Voltage Cells—*A. Daniell's Cell.*—In order to avoid the effect of polarisation of the positive plate (*i.e.*, the copper plate in the simple cell of § 136) and consequent diminution of the effective e. m. f. of the cell, various constructions of cells have been devised, of which the earliest was the Daniell's cell. For special purposes this cell is still in use. The negative pole is a zinc rod which dips in zinc sulphate or water contained in a porous pot. The porous pot is placed in a second larger vessel, which contains copper sulphate, and a copper plate dipped in the copper sulphate forms the positive pole. In some cases the outer containing vessel is itself of copper, and forms the positive pole.

When the cell supplies a current the chemical action which results causes the zinc rod to dissolve and to form zinc sulphate in the porous pot. The zinc sulphate diffuses through the porous pot, and decomposes the copper sulphate, and copper becomes deposited on the copper plate instead of hydrogen, as was the case in the simple cell.

The copper sulphate in the outer vessel thus becomes mixed with zinc sulphate, but, owing to the fact that the copper in copper sulphate has a less affinity for the radical SO_4 than zinc, only copper is deposited on the copper plate.

Let E volts be the e. m. f. of the cell—that is, the p. d. of the plates—when no current flows in the cell. If the e. m. f. E remains constant when a current flows, then for every coulomb of quantity of electricity which passes round the circuit the chemical action of the cell must supply the energy E ergs, since this is the energy necessary to transfer 1 coulomb of positive electricity from one conductor to another at a potential E volts higher.

This transfer takes place in the cell, and the energy is then transformed into heat in the conductors forming the circuit of the cell, when the terminals are connected together by a wire (see also § 131).

The energy released by the chemical action may be calculated as follows: The chemical action in the cell may be represented by the equations—

In porous pot:

$$\underset{\text{Zinc}}{Zn} + \underset{\text{Sulphuric acid}}{H_2SO_4} = \underset{\text{Zinc sulphate}}{ZnSO_4} + \underset{\text{Hydrogen}}{H_2}.$$

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* See also survey of the subject "On the Seat of the Electromotive Force in a Voltaic Cell," by Sir Oliver Lodge, F.R.S., *Journal of Inst. of Elec. Eng.*, 1885.

The liberated hydrogen diffuses through the porous pot and decomposes the copper sulphate.

In outer vessel:

$$\underset{\text{Copper sulphate}}{CuSO_4} + \underset{\text{Hydrogen}}{H_2} = \underset{\text{Sulphuric acid}}{H_2SO_4} + \underset{\text{Copper}}{Cu},$$

the copper being deposited on the copper plate.

That is to say, when one atom of zinc (which weighs 65·4 units) dissolves from the zinc rod, one atom of copper (which weighs 63·6 units) is deposited on the copper plate.

Experiment shows that when 1 gm. of zinc is converted into zinc sulphate, the energy *released* is equivalent to 1670 calories or 698×10^8 ergs, and when 1 gm. of copper is separated out from copper sulphate the energy *absorbed* is equivalent to 909 calories or 380×10^8 ergs. [*Note.*—1 calorie is the heat necessary to raise 1 gm. of water through 1° C., and is equal to $4{\cdot}18 \times 10^7$ ergs.]

A further experimental fact is that, in order to dissolve 1 gm. of zinc, a current of 1 ampère must pass for 2958 seconds—*i.e.*, 2958 coulombs of electricity must pass through the cell (see § 162).

Hence, if 1 coulomb of electricity passes through the cell, $\frac{1}{2958}$ gm. of zinc is dissolved, and $\frac{63{\cdot}6}{65{\cdot}4} \times \frac{1}{2958}$ gm. of copper is deposited on the copper plate.

Hence, when 1 coulomb passes through the cell, 0·000338 gm. of zinc is dissolved and 0·000329 gm. of copper deposited. The energy *released* by the chemical action when 1 coulomb passes is therefore—

$$0{\cdot}000338 \times 698 \times 10^8 \text{ ergs}$$
$$= 2{\cdot}36 \times 10^7 \text{ ergs.}$$

The energy *absorbed* is—

$$0{\cdot}000329 \times 380 \times 10^8 \text{ ergs}$$
$$= 1{\cdot}25 \times 10^7 \text{ ergs.}$$

The resultant energy which is available for sending 1 coulomb round the circuit is therefore—

$$1{\cdot}11 \times 10^7 \text{ ergs.}$$

If E volts is the e. m. f. of the cell, the electrical energy dissipated in sending 1 coulomb round the circuit is $E \times 10^7$ ergs (see §§ 8 and 28).

Hence $$E \times 10^7 = 1{\cdot}11 \times 10^7;$$

that is, the calculated value of the e. m. f. of the Daniell's cell is 1·11 volts.

Actually the e. m. f. of the Daniell's cell is about 1·07 volts, and the difference is to be taken as due, at least in part, to thermo-electric effects (see § 176) which develop at the junctions of plates and liquid in the cell. The e. m. f. of the cell depends to some extent on the concentration of the liquid.

It may be observed here that the numbers which represent the weights of the respective metals which take part in the chemical action when 1 coulomb flows are termed the *electro-chemical equivalents* of the metals. This is more fully dealt with in Chapter XII.

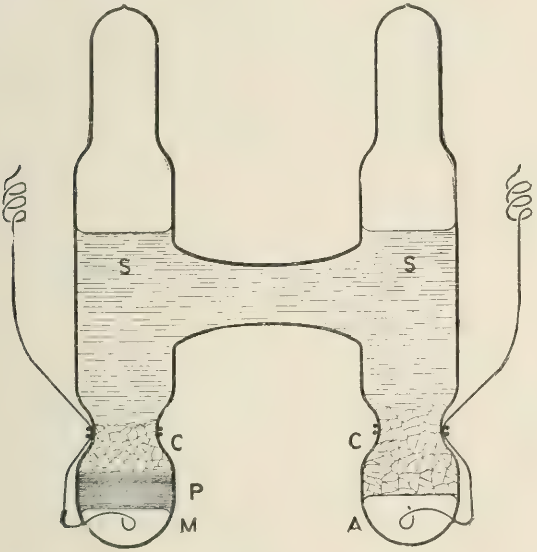

Fig. 170.

M = Mercury; P = mercurous sulphate paste; C = crystals of cadmium sulphate; S = saturated solution of cadmium sulphate; A = cadmium amalgam.

B. Weston Norman Cell.—This cell was recommended as the standard by the International Conference on Electrical Standards in 1908. The construction of the cell is shown in Fig. 170. The high state of perfection reached in this type of cell is shown by the fact that, if great care is taken in obtaining pure and uniform chemicals, such cells can be set up by different experimenters with different materials, and the e. m. f. of the cells will agree within a few parts in 100000. The e. m. f. of the standard Weston Normal Cell is 1·0183 volts at a temperature of 20° C.

It is one of the most valuable features of this type of cell that the variation of the e. m. f. with temperature is extremely small. The internal resistance of the cell is about 900 ohms. If the cell is short-circuited it has a remarkable recuperative power after the short-circuit is removed. Thus a cell short-circuited for five minutes was one part in 1000 low in e. m. f. one minute afterwards, but completely recovered in one and a half hours.

138. Leclanché Cell.—This cell is in very common use for a variety of purposes in which it is only desired to obtain a current for a brief interval of time. In this cell an amalgamated zinc rod in a saturated solution of ammonium chloride (sal-ammoniac) forms the negative pole, and a carbon plate closely packed with fragments of black oxide of manganese forms the positive plate. The e. m. f. of the cell is about 1·4 volts on open circuit, but falls considerably when a current passes, owing to polarisation of the carbon plate. If the current is stopped, the cell quickly recuperates, and the plates may be left in contact with the liquid for long periods without injury.

CHAPTER XI

ELECTRIC RESISTANCE

139. Measurement of Electric Currents.—An electric current is measured by means of its magnetic or heating effects (see §§ 200 and 221), and an instrument which indicates the magnitude of the current passing through it is termed *a galvanometer* when the current is to be measured with the greatest possible accuracy, or an *ammeter* when the measurement is to be made with a degree of accuracy sufficient for practical purposes.

140. Ohm's Law.—Suppose a number of Daniell cells are connected in series, and the positive terminal of the battery connected through a galvanometer to the terminal B and the negative terminal of the battery to the terminal A (Fig. 171). If A and B are joined by a wire, a current will flow round the circuit, and experiment shows that the current is the same at every part of the circuit. Now connect a quadrant electrometer to the ends of the wire BA, and note the p. d. between these points and at the same time the current indicated by the galvanometer G. If a series of such readings be taken, say, by varying the number of the cells in the battery, it will be found that the ratio

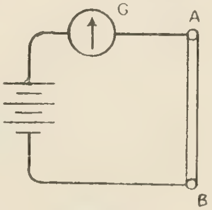

Fig. 171.

$$\frac{\text{potential difference}}{\text{current}}$$

is constant, whatever the value of the current may be, *provided* that the physical conditions (*i.e.*, temperature and strain of the wire) remain unaltered whilst the measurements are being made. That is to say, if V is the potential difference between the ends of the wire when the current I is flowing in the wire, the ratio

$$\frac{V}{I}=R,$$

where R is a constant and is termed the **resistance** of the wire.

The constancy of this ratio of p. d. and current is known as **Ohm's Law.** If the p. d. is measured in volts, and the current in ampères, the resistance is given in **ohms.**

Ohm's Law applies to liquid conductors (electrolytes) as well as to solid conductors, and its accuracy has been established by the most searching experimental tests in which a deviation of one part in 100000 could have been detected, but no departure from the constancy of the ratio of V to I has been observed, if the conditions as to constancy of physical state during the measurements are fulfilled.

Now suppose the whole circuit of the battery be considered, and let E represent the e. m. f. of the battery—*i.e.*, the p. d. between the terminals when no current is flowing. Then if a current I passes through the circuit

$$\frac{E}{I}=R,$$

where R is the **resistance of the whole circuit, including the liquid conductor of the battery.**

In the case of a simple cell it is to be observed that as the copper plate becomes polarised, an e. m. f. e is developed at the surface of contact of the liquid and the copper plate which is opposed to the main e. m. f. E, hence the current is given by the expression—

$$\frac{E-e}{I}=R.$$

Hence, Ohm's Law may be stated generally as follows:

$$\frac{\text{Total e. m. f. in the circuit}}{\text{Current in the circuit}}=\text{total resistance of the circuit.}$$

The great importance of Ohm's Law is that it signifies that the resistance of a conductor is independent of both the p. d. and the current in the conductor, and it forms a basis for the whole system of electrical measurements (see Section VI.).

A steady current in a circuit denotes a steady e. m. f. acting in the circuit, and a steady e. m. f. in the circuit denotes that a steady current flows. Ohm also showed that the resistance of a conductor of uniform cross-section S and length l is given by the expression—

$$R=\rho\,\frac{l}{S},$$

where ρ is a constant characteristic of the material of the conductor, and is called the **specific resistance** or **resistivity.**

If the section of the conductor is not uniform throughout its length, the resistance is given by—

$$R=\rho\Sigma\,\frac{\delta l}{S},$$

where S is the section considered uniform over the element of length δl and the summation taken for the whole length of the conductor.

Ohm's Law does not apply in the case of a current passed through a rarefied gas.

141. Unit of Resistance: Definition of the Ohm.

The international ohm is the resistance offered to an unvarying electric current by a column of mercury at the temperature of melting ice, 14·45 gms. in mass, of constant cross-sectional area, and of length 106·3 cms.

One method by means of which this determination of the value of the ohm has been made is referred to on p. 479, Section VI.

The above specification of the ohm is intended to represent a resistance equal to 10^9 absolute electromagnetic c. g. s. units, the absolute electromagnetic c. g. s. unit of resistance being such that 1 electromagnetic unit of current flows in the resistance when 1 electromagnetic unit of p. d. is maintained across the terminals (see Section VI.).

Multiples and submultiples of the ohm are used in practice—viz.: 1 megohm $=10^6$ ohms. 1 microhm $=10^{-6}$ ohms.

142. Specific Resistance.—In the formula given in § 140 defining specific resistance, if a conductor is chosen of length $l=1$ cm. and cross-section $S=1$ sq. cm., then—

$$R=\rho.$$

Hence the specific resistance of a conducting material is the resistance between opposite faces of a cube of the material of 1 cm. sides. More briefly expressed, the specific resistance is the resistance per centimetre cube of the material.

Sometimes the specific resistance is given in terms of the inch as unit of length, and the square inch as unit of cross-sectional area, and briefly expressed as the resistance per inch cube of the material.

Knowing the specific resistance of the material of a conductor, it is easy to calculate the resistance of the conductor of a given length and cross-sectional area.

The **conductivity** of a conductor is the reciprocal of the specific resistance (or resistivity), and is measured in *mhos* per centimetre cube of the material. Thus, if the specific resistance is ρ ohms per centimetre cube the conductivity is $\frac{1}{\rho}$ mhos per centimetre cube.

Copper is of special importance in electrical use on account of its high conductivity, and electrical conductors are almost exclusively made of copper, although for some purposes, such as overhead transmission lines, aluminium is often used.

Some of the earliest and most careful measurements of the specific resistances of various metals were made by Matthiessen (1831–1870), and his values for hard-drawn and annealed commercial copper respectively are still the standard values, and are often referred to as *Matthiessen's Standards*—viz.:

The standard of resistance for hard-drawn high conductivity commercial copper is a wire of pure hard-drawn copper 1 *metre long, weighing* 1 *gm., of which the resistance is* 0·153858 *international ohm at* 60° *F.* (*i.e.*, 15·6° *C.*).

The specific gravity of copper varies from 8·89 to 8·95 at 0° C. The standard value for high conductivity commercial copper is 8·912 at 0° C., which corresponds to a weight of 555 lbs. per cubic foot (*i.e.*, 8·89 gms. per c. cm.) at 60° F. (*i.e.*, 15·6° C.).

Matthiessen found that the conductivity of annealed copper is greater than that of hard-drawn copper by about 2·25 per cent. to 2·5 per cent.

The standard of resistance for annealed commercial copper is a wire of pure annealed copper 1 *metre long, weighing* 1 *gm., whose resistance is* 0·15082 *international ohm at* 60° *F.* (*i.e.*, 15·6° *C.*).

The following table gives the values of the specific resistance of some of the substances commonly used in electrical engineering practice:

Substance.	*Specific Resistance in Ohms per Centimetre Cube at 0° C.*	*Density.*
Aluminium (hard)	$2·8 \times 10^{-6}$	2·7
High conductivity commercial copper (annealed)	$1·594 \times 10^{-6}$	8·9
Copper (hard-drawn)	$1·62 \times 10^{-6}$	
Iron (annealed)	$10·57 \times 10^{-6}$	7·8
Cast steel	20×10^{-6}	7·8
Cast iron	100×10^{-6}	7·2
Alloyed dynamo iron	$42·5 \times 10^{-6}$	7·8
Carbon (arc lamp)	4×10^{-3} to 8×10^{-3}	1·9 to 2·3
Carbon (filament)	5×10^{-3}	
Graphite	$0·5 \times 10^{-3}$	
Copper sulphate at 18° C. (saturated solution)	25	—
Zinc sulphate at 18° C. (saturated solution)	28	—
Sulphuric acid at 18° C.:		
Diluted 5% acid	4·88	1·04
,, 10% ,,	2·56	1·07
,, 20% ,,	1·56	1·14
,, 25% ,,	1·39	1·17
,, 75% ,,	6·57	1·7
Water (ordinary distilled)	about $0·5 \times 10^{6}$	1·0
Sea water	about 100	1·025

The specific resistances of some of the pure metals are given in Table I., p. 199.

The specific resistance of mercury is of great importance in connection with the specification of the standard ohm (§ 141).

The resistance of the element *selenium* is affected by light, and is a better conductor in the light than in the dark. Various practical applications of this special property have been developed.

The resistance of the metal *bismuth* is increased when placed in a magnetic field.

If the magnetic field is transverse to a rod of bismuth the increase of resistance is far greater than when the direction of the magnetic field is along the rod.

The following table gives the resistance of a bismuth wire kept at constant temperature (19° C.) and placed in a transverse magnetic field (Henderson, *Phil. Mag.*, 1894).

Intensity H of Transverse Magnetic Field in C. G. S. Units.	*Resistance of Bismuth Wire.*
0	1
12500	1·63
27450	2·54
38900	3·34

At – 185° C. a transverse magnetic field of intensity H = 21800 increases the resistance of a bismuth wire 150 times (Dewar).

The increase of resistance of a bismuth wire with increasing intensity of magnetic field has been used as a means of exploring a magnetic field.

The specific resistance of alloys is, generally speaking, much higher than that of pure metals. The values for some of the alloys are given in Table II., p. 199.

Generally speaking, an alloy having a high specific resistance is poor mechanically.

When a pure metal is alloyed with a small proportion of another metal, the specific resistance of the alloy is widely different, according to the nature of the metals used. Thus an alloy of pure copper with 3 per cent. aluminium has a specific resistance about 5½ times that of copper.

The presence of a very small proportion of a non-metallic element in an otherwise pure metal has a very great effect in increasing the specific resistance of the metal; pure iron with 0·01 per cent. C has a specific resistance of about 10 microhms per cm. cube, whereas an addition of 3·4 per cent. silicon increases the specific resistance about 4½ times (see p. 140). Certain metallic elements also show the same effect; thus platinoid has a specific resistance 30 per cent. greater than that of German silver, although it differs from it merely in containing a trace of tungsten.

The specific resistance of non-metallic conductors is in all cases higher than that of any pure metal.

Special alloys are manufactured of which the values of the specific resistances are exceptionally large. Such alloys are used when it is desired to consume as much power as possible for a minimum amount of resistance material (*e.g.*, for electric furnaces, etc.).

A well-known alloy used for such purposes is nickel-chrome, and some data of this material are given in the following table:

Size S.W.G.	*Diameter, Inch.*	*Resistance in Ohms per* 1000 *Yards.*	
		200° *C.*	400° *C.*
16	0·064	451	494
18	0·048	800	879
20	0·036	1423	1590
22	0·028	2354	2583

The carrying capacity of nickel-chrome for well-ventilated open spirals is given in the following table:

Size S.W.G.	*Rise in Temperature.*		*Size. S.W.G.*	*Rise in Temperature.*	
	100° *C. Ampères.*	200° *C. Ampères.*		100° *C. Ampères.*	200° *C. Ampères.*
22	1·6	2·1	14	6	10
20	2·0	3·2	12	10	15
18	3·0	4·2	10	16	24
16	4·2	7·0	8	22	36

Another well-known high-resistance material is "Eureka." For this alloy the temperature coefficient of the resistance increase is practically nil.

Size S.W.G.	*Diameter (Inch).*	*Ohms per* 1000 *Yards.*	*Size S.W.G.*	*Diameter (Inch).*	*Ohms per* 1000 *Yards.*
22	0·028	1093	14	0·08	133·9
20	0·036	661·3	12	0·104	79·3
18	0·048	371·8	10	0·128	52·3
16	0·064	209·4	8	0·16	33·5

The carrying capacity of "Eureka" for well-ventilated open spirals is given in the following table:

Size S.W.G.	*Rise in Temperature.*		*Size S.W.G.*	*Rise in Temperature.*	
	100° C. Ampères.	*200° C. Ampères.*		*100° C. Ampères.*	*200° C. Ampères.*
22	2·2	3·2	14	9·5	15
20	3·0	4·7	12	16·8	24
18	4·3	7·0	10	22·8	36
16	6·0	10·4	8	33·0	52

The specific resistance of liquids is, generally speaking, much higher than that of any solid conductor.

One of the best conducting liquids is fused lead chloride, which has a specific resistance of about 370000 microhms per cm. cube, whereas the specific resistance of any alloy is seldom greater than 100 microhms per cm. cube.

The specific resistance of dielectrics, or so-called insulators, is given in the following table (see also p. 216):

Substance.	*Specific Resistance in Ohms per Cm. Cube.*	*Temperature in ° C. to which the Values refer.*
Bohemian glass	61×10^{12}	60°
Mica	84×10^{12}	20°
Gutta-percha	$7{,}000 \times 10^{12}$	0°
	450×10^{12}	24°
Flint glass	$1{,}020 \times 10^{12}$	60°
Vulcanised indiarubber	$1{,}500 \times 10^{12}$	15°
Shellac	$9{,}000 \times 10^{12}$	28°
Pure indiarubber	$10{,}900 \times 10^{12}$	24°
Ebonite..	$28{,}000 \times 10^{12}$	46°
Paraffin	$34{,}000 \times 10^{12}$	46°
Linseed oil	5×10^{10}	18°
Olive oil..	1×10^{12}	18°

It is to be observed that the majority of metals, when in a finely-powdered condition, are practically non-conductors: a mass of metallic powder or filings may be made to pass suddenly into a conducting state, however, by being exposed to the influence of an electric wave. The same is true of the loose contact of two metallic conductors. Thus, if a steel point presses very lightly against a metallic plate of, say, aluminium, it is found that this metallic contact, if carefully adjusted, is non-conducting, but if an electric wave is generated in the neighbourhood the contact suddenly becomes conducting. These properties were utilised formerly as detectors for receiving wireless telegraphy signals.

Certain dry chemical compounds can be changed with extreme rapidity from very good conductors of electricity to almost perfect insulators by the application of a slight degree of heat—*e.g.*, lead peroxide (PbO_2). Lead peroxide has a specific resistance of about 1 ohm per inch cube, the resistance varying with the pressure with which it has been compressed. At a temperature of about 150° C. lead peroxide is reduced to red lead (Pb_3O_4), which has a specific resistance of about 24×10^6 ohms per inch cube. At slightly higher temperatures red lead reduces to litharge (PbO), which is practically an insulator. This property is made use of in one form of lightning arrester (*General Electric Review*, September, 1918).

143. Influence of Temperature on Resistance: Temperature Coefficients.—It is found that the resistance of all pure metals and of nearly all alloys *increases* with rise of temperature.

The resistance of carbon and of non-metallic liquids (*i.e.*, electrolytes) *decreases* with rise of temperature.

The variation of resistance with temperature of metals and alloys is closely represented by the following formula for a wide range of temperatures:

$$R_t = R_o (1 + \alpha t),$$

where R_t is the resistance at the temperature t
,, R_o ,, ,, ,, ,, 0°,

and α is a coefficient, which is a characteristic of the material of the conductor, and is termed the *temperature coefficient.*

In Table I., column 3 (p. 199), are given the values of α for pure metals.

In Table II., column 3 (p. 199), are given the values of α for alloys.

The resistance of carbon *decreases* with rise of temperature, the value of the temperature coefficient a being from about $-0{\cdot}0005$ to $-0{\cdot}0008$.

The specific resistance of iron changes in a remarkable way in the neighbourhood of the critical temperature 780° C. (see Section II., § 119).

From some experiments by Hopkinson and Morris* it was found that the mean temperature coefficient of iron near 0° C. was 0·0057. At 765° C. the temperature coefficient rose to a maximum of 0·0204, and at 1000° C. fell to 0·00244.

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* *Phil. Mag.*, September, 1897.

TABLE I

RESISTANCE DATA OF PURE METALS. (FLEMING AND DEWAR)

Metal (Pure).	*Resistance at 0° C. in Ohms per Centimetre Cube.*	*Mean Temperature Coefficient α between 0° C. and 100° C.*
Silver (electrolytic and well annealed)	$1{\cdot}468 \times 10^{-6}$	0·00400
Copper (electrolytic and well annealed)	$1{\cdot}561 \times 10^{-6}$	0·00428
Gold (annealed)	$2{\cdot}197 \times 10^{-6}$	0·00377
Aluminium (annealed)	$2{\cdot}665 \times 10^{-6}$	0·00435
Zinc	$5{\cdot}751 \times 10^{-6}$	0·00406
Nickel (electrolytic)	$6{\cdot}935 \times 10^{-6}$	0·00618
Iron (annealed)	$9{\cdot}065 \times 10^{-6}$	0·00625
Platinum (annealed)	$10{\cdot}917 \times 10^{-6}$	0·003669
Tin (pressed)	$13{\cdot}048 \times 10^{-6}$	0·00440
Lead (pressed)	$20{\cdot}38 \times 10^{-6}$	0·00411
Bismuth (electrolytic)	110×10^{-6}	0·00433
Mercury (pure)	$94{\cdot}07 \times 10^{-6}$	0·00098

TABLE II

RESISTANCE DATA OF ALLOYS. (FLEMING AND DEWAR)

Alloy.	*Resistance at 0° C. in Ohms per Centimetre Cube.*	*Mean Temperature Coefficient taken at 15° C.*	*Composition of Alloy in Percentages.*
Platinum silver	$31{\cdot}582 \times 10^{-6}$	0·000243	Pt 33; Ag 66
Platinum iridium	$30{\cdot}896 \times 10^{-6}$	0·000822	Pt 80; Ir 20
Gold-silver	$6{\cdot}28 \times 10^{-6}$	0·00124	Au 90; Ag 10
Manganese steel	$67{\cdot}148 \times 10^{-6}$	0·00127	Mn 12; Fe 78
Nickel steel	$29{\cdot}452 \times 10^{-6}$	0·00201	Ni 4·35
German silver	$29{\cdot}982 \times 10^{-6}$	0·000273	$Cu_5Zn_3Ni_2$
Platinoid[*]	$41{\cdot}731 \times 10^{-6}$	0·00031	—
Manganin	$46{\cdot}678 \times 10^{-6}$	0·0000	Cu 84; Mn 12; Ni 4

144. Resistance of Conductors at very Low Temperatures: Super-Conductors.—Professor H. K. Onnes,[†] of Leyden, has recently made some very interesting experiments on the behaviour of conductors at very low temperatures.

It has been deduced from theoretical considerations that the specific resistance of mercury at a temperature of 4·25° absolute (*i.e.*, $-268{\cdot}75°$ C.) would still be measurable, but at 2° absolute ($-271°$ C.) would become almost indefinitely small. Onnes, however, has found that at a temperature of 4·19°

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[*] Platinoid is similar in composition to German silver, but contains a trace of tungsten.

[†] *Electrician*, November 15, 1918.

absolute the value of the specific resistance falls suddenly to an almost immeasurably small value. This critical temperature marks the transition from the conditions of normal conductivity to "super-conductivity." When the temperature had fallen below the critical temperature it was found possible to pass a current at a density of 1000 *ampères per square millimetre* through a conductor 1 metre long without any appreciable difference of potential between the ends.

By using 1000 turns of lead wire (critical temperature 6° absolute), 0·1 mm. thick, it was possible to obtain an induction density in air of 100000 lines per square cm.

By the use of a lead wire spiral with the two ends welded together by the oxyhydrogen blowpipe, the spiral was found to have a resistance of one twenty-billionth of an ohm when the temperature was reduced below the critical value, whereas at the normal temperature of the room the resistance of the spiral was 736 ohms. In this way a current of about 0·6 ampère excited in the spiral lasted for several days.

145. Energy of the Electric Current: Joule's Law.—By definition (see § 28) the potential difference between two points A and B is the work done by the electric force when 1 unit of electricity moves from the point A at the higher potential to the point B at the lower potential.

If the p. d. between the points A and B is 1 electrostatic unit, the work done in moving 1 electrostatic unit of quantity from A to B is 1 erg.

If the p. d. between A and B is V volts, and the quantity of electricity moved from A to B is Q coulombs, the work done is—

$$3 \times 10^9 \frac{V}{300} Q \text{ ergs} = VQ \times 10^7 \text{ ergs},$$

since 300 volts = 1 electrostatic unit of p. d., and 1 coulomb = 3×10^9 electrostatic units of quantity (§§ 8 and 28).

If the points A and B are connected by a conductor and a current of I ampères—that is, I coulombs per second flows in the conductor—the work done by the electric force is—

$$VI \times 10^7 \text{ ergs per second.}$$

But by Ohm's Law $V = IR$, where R is the resistance in ohms of the conductor connecting A and B. Hence the work done by the electric force is—

$$\begin{aligned} &I^2R \times 10^7 \text{ ergs per second} \\ &= I^2R \text{ watts,} \end{aligned}$$

1 **watt** being defined as 10^7 ergs per second.

This is the energy supplied by the system which keeps the

Dec 30/25

p. d. of the points A and B at V volts. For example, in a battery the work is obtained from the chemical energy of the battery (see § 137).

The question then arises as to what becomes of this energy, and Joule made a classical series of experiments (about the year 1843) to decide this question. Joule showed *that if a current of I ampères flows in a conductor of R ohms resistance for* t *seconds, the heat developed in the conductor is*—

$$H = 0{\cdot}239\, I^2Rt \text{ calories.}$$

This result is Joule's Law. It states that the heat produced in a conductor in a given time is proportional to the square of the current strength and to the resistance of the conductor.

Joule also showed that 1 calorie was equivalent to $4{\cdot}18 \times 10^7$ ergs, and hence the heat developed in the conductor in t seconds is equivalent to—

$$I^2Rt \times 10^7 \text{ ergs;}$$

or

$$I^2R \times 10^7 \text{ ergs per second.}$$

Now the quantity 10^7 ergs is termed 1 *joule*, and hence the heat developed in the conductor is—

$$I^2R \text{ joules per second;}$$

that is—

$$I^2R \text{ watts,}$$

since 10^7 ergs per second (*i.e.*, 1 joule per second) is defined as 1 watt.

It was thus demonstrated that the whole of the energy corresponding to the passage of electricity between two points in a conductor is converted into heat.

146. Generalised Statement of Joule's Law.—In the example given in § 145 the current in the conductor AB was supposed to be supplied from a source of electricity such as a battery, and there was assumed to be no source of e. m. f. in the conductor between the points A and B.

It is possible, however, to convert the electrical energy of the battery directly into mechanical energy, as in the case in which an electric motor is connected in circuit between the points A and B (see § 200). In this case an opposing or *back e. m. f.* developed by the consuming device and the energy supplied is—

$$VI = eI + I^2R,$$

where e is the back e. m. f. developed by the consuming device and R is the resistance. The mechanical energy generated is then equal to eI watts.

Eventually, however, whatever form the consuming device may take, the energy is dissipated in heat.

147. Fuses.—When a current flows through a wire the temperature of the wire rises and would continue to rise indefinitely if no heat were lost to the surrounding medium. The temperature which is actually reached is that at which the rate of loss of heat to the surrounding medium is equal to the rate at which the heat is developed in the wire by the electric current. By Newton's Law of cooling, the loss of heat by radiation per unit of area at each instant is proportional to the excess of temperature of the wire above the surrounding medium.

Let l cms. be the length of the wire.
,, d cms. be the diameter of the wire.
,, ρ be the specific resistance in ohms per cm. cube.
,, R ohms be the resistance.
,, I ampères be the current.
,, θ be the temperature rise above the surrounding medium.
,, λ be radiation constant.

Heat generated $=I^2R$ joules per second.

$=0{\cdot}24I^2R$ calories per second.

Heat radiated $=\pi d l \lambda \theta$.

Hence

$$\pi d l \lambda \theta = 0{\cdot}24 I^2 R$$

$$= \frac{0{\cdot}24 I^2 \cdot l}{\frac{\pi}{4} d^2} \rho;$$

or

$$I^2 = \frac{\pi^2}{4} \frac{d^3 \lambda \theta}{0{\cdot}24 \rho} = 10{\cdot}2 d^3 \frac{\lambda}{\rho} \theta;$$

or

$$I = k d^{\frac{3}{2}} \theta^{\frac{1}{2}},$$

where k is a constant.

Hence, for a wire of a given material the current which is necessary to fuse the wire (fusing temperature, θ) is proportional to the diameter of the wire raised to the power $\frac{3}{2}$.

Actually, however, the practical fuse problem is not quite so simple since the fixture of the ends of the wire contribute to the dissipation of the heat, and in cases where two or more wires are adjacent the mutual heating effect of the two wires influences the result. The formula deduced above is therefore only to be taken as a general indication of the way in which the fusing current varies with the diameter of the wire.

148. Discharge of a Condenser.—Suppose a condenser is charged to a p. d. of V_o volts with a quantity Q_o coulombs—that is, the capacity of the condenser is

$$C = \frac{Q_o}{V_o} \text{ farads}$$

Now let the plates of the condenser be connected by a wire of resistance R ohms. A current will flow in the wire, and if i, q, and v be corresponding values at any instant of current, quantity, and p. d. respectively—

$$i = -\frac{dq}{dt} = -C\frac{dv}{dt} \text{ ampères.}$$

But $$i = \frac{v}{R} \text{ ampères.}$$

Hence $$\frac{v}{R} = -C\frac{dv}{dt};$$

or $$dt = -CR\frac{dv}{v},$$

from which it follows that the p. d. after time t seconds from the moment at which the plates are joined by the wire is—

$$V_t = V_o e^{-\frac{t}{CR}}.$$

In Fig. 172 the values of V_t are plotted as a function of the time for the case in which $V_o = 1000$ volts: $C = 10$ mfd.: $R = 1000$ ohms.

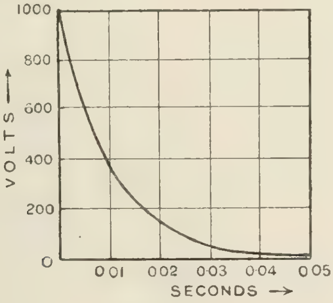

FIG. 172.

Further, since $$V_t = V_o e^{-\frac{t}{CR}};$$

i.e. $$CR = -\frac{t}{\log_e \frac{V_t}{V_o}};$$

or $$R = \frac{t}{C \log_e \frac{V_o}{V_t}},$$

it is possible to determine the resistance R when observations of t, V_o, and V_t are taken, assuming the capacity C to be known.

This result may be used, for example, to determine the insulation resistance of a cable.

149. Kirchoff's Rules.—The following two deductions from Ohm's Law are very useful in dealing with a network of conductors.

1. *If several conductors meet at a point the algebraic sum of the currents flowing to the point is zero, currents flowing to the point being reckoned positive and currents flowing from the point being reckoned negative.*

This is merely a statement of the fact that there is no accumulation of electricity at any point in a circuit in which a current is flowing, and this result is really included in Ohm's Law (§ 140).

Thus, in Fig. 173—

$$i_1 - i_2 - i_3 - i_4 = 0;$$

or

$$i_1 = i_2 + i_3 + i_4.$$

This result is useful in dealing with alternating current systems (see p. 357, Section V.).

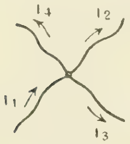

Fig. 173.

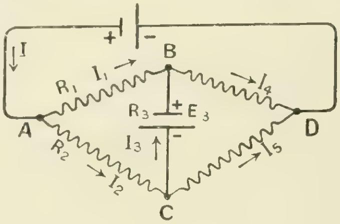

Fig. 174.

2. *If two or more conductors form a closed figure, the algebraic sum of the products of the resistance and current taken for each conductor (the current being reckoned positive the same way round the figure) is equal to the algebraic sum of the e. m. f.'s acting round the figure in the same direction—that is,*

$$\Sigma I R = \Sigma E,$$

where the summation is to be taken round the closed figure.

This result may be deduced from Ohm's Law (§ 140), and will be clear from the following example.

Suppose the arrangement of circuits is as shown in Fig. 174, and consider the mesh ABCA. From Kirchoff's rule—

$$E_3 = I_3R_3 + I_2R_2 - I_1R_1.$$

Now $E_3 - I_3R_3$ is the amount by which the potential of B is *above* the potential of C, and I_2R_2 is the amount by which the potential of C is *below* the potential of A. Hence $I_2R_2 - (E_3 - I_3R_3)$ is the amount by which the potential of B is *below* the potenital of A. But I_1R_1 is the amount by which the potential of B is *below* the potential of A.

Hence $$I_1R_1=I_2R_2-E_3+I_3R_3;$$
or $$E_3=I_3R_3+I_2R_2-I_1R_1.$$

If a loop consists of three conductors, as in Fig. 175, in which no e. m. f. acts, then ΣIR for the closed loop is zero.

An example of the application of this result is found in connection with mesh connected alternating current systems (see p. 373, Section V.).

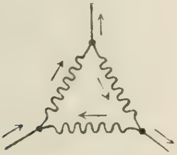

Fig. 175.

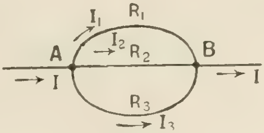

Fig. 176.

150. Resistance of Conductors connected in Series or Parallel Arrangement—*A. Series Connection.*—If several conductors are connected end to end—that is, *in series*, so that the same current flows through each conductor—the resistance of the combination is the ratio of the p. d. at the terminals of the series to the current through the series. Since the p. d. at the terminals is the sum of the p. d.'s at the ends of the individual conductors (§ 28), the resistance of the series is the sum of the resistances of the individual conductors. Thus, if $R_1 : R_2 : R_3$ . . . are the resistances of the individual conductors, the total resistance of the series is—

$$R=R_1+R_2+R_3+\ .\ .\ .$$

B. Parallel Connection.—If a number of conductors are joined so that one end of each conductor is connected to a common point A, and the other end of each conductor to a second common point B, the conductors are said to be connected *in parallel.*

Let $R_1 : R_2 : R_3$ in Fig. 176 be connected in parallel between the points A and B, and let a current flow from A to B through the parallel arrangement. If V volts is the p. d. between the points A and B when a current I ampères at A enters and leaves at B, the resistance of the parallel connection is—

$$R=\frac{V}{I}\text{ ohms.}$$

Let $I_1 : I_2 : I_3$ respectively be the currents through the individual branches.

Then $$I=I_1+I_2+I_3.$$

But $$I=\frac{V}{R} : I_1=\frac{V}{R_1} : I_2=\frac{V}{R_2} : I_3=\frac{V}{R_3}.$$

Hence $$\frac{V}{R}=V\left[\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right];$$

or $$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}.$$

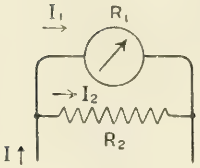

Fig. 177.

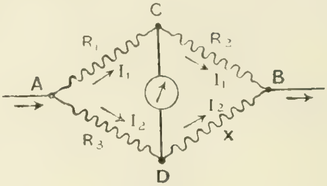

Fig. 178.

A specially important example is that in which there are only two branches in parallel, such as when a *shunt* is used with a galvanometer (Fig. 177). Then the equivalent resistance is given by—

$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2};$$

or $$R=\frac{R_1R_2}{R_1+R_2}.$$

In this case $$\frac{I_1}{I}=\frac{V}{R_1}\times\frac{R}{V}=\frac{R}{R_1};$$

or $$\frac{I_1}{I}=\frac{R_2}{R_1+R_2},$$

$$I_1=I\,\frac{R_2}{R_1+R_2}.$$

Also $$I_1=I_2\,\frac{R_2}{R_1}.$$

Thus, if R_2 is the resistance of the shunt (Fig. 177), and R_1 the resistance of the galvanometer, the current through the galvanometer is $\frac{R_2}{R_1+R_2}$ of the total current in the circuit, or the current through the galvanometer is $\frac{R_2}{R_1}$ times the current through the shunt.

151. Wheatstone Bridge.—In Fig. 178 the two points A and B are connected by the branches ACB and ADB, of which ACB consists of the resistances R_1 and R_2 in series, and ADB the resistances R_3 and X in series.

Suppose a current enters at A and leaves at B, dividing between the two branches ACB and ADB.

The potential of the point C will be something intermediate between that of A and that of B. It will clearly be possible to find some point D in the branch ADB, such that the potential of D is the same as that of C. If such a point be found and a galvanometer be connected to the points C and D—that is, to *bridge* the two branches—there will be no deflection of the galvanometer. This condition will be satisfied if—

$$I_1R_1=I_2R_3, \text{ and } I_1R_2=I_2X;$$

that is—

$$\frac{I_1}{I_2}=\frac{R_3}{R_1}=\frac{X}{R_2};$$

or

$$X=\frac{R_2}{R_1}R_3.$$

If $R_1=R_2 \quad : X=R_3.$

,, $R_1=10R_2 \quad : X=\frac{1}{10}R_3.$

,, $R_1=100R_2 : X=\frac{1}{100}R_3.$

If the values of $R_1:R_2:R_3$ are known and adjustable, the value of the unknown resistance X can be determined with great accuracy.

The above description gives the principle of this method of measuring resistances.

Actually the Wheatstone Bridge is made in a variety of forms, the resistances $R_1:R_2:R_3$ being built up in one box which sometimes also includes the galvanometer G. If very great accuracy is required, however, the galvanometer used is of the reflecting mirror type (see § 208).

152. Best Arrangement of Cells.—Suppose there are N similar cells, each of internal resistance R and of e. m. f. E, and suppose it is desired to use these cells to send a current through a given resistance R_1. It is required to find what arrangement of the cells will give the maximum current in the resistance R_1.

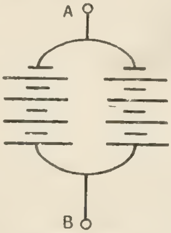

FIG. 179.

Suppose the cells be arranged X cells in series and Y groups in parallel of X cells (Fig. 179).

Then $XY=N.$

The current in R_1 will be—

$$\frac{XE}{\frac{XR}{Y}+R_1}=\frac{YXE}{XR+YR_1}=\frac{NE}{XR+YR_1}.$$

This will be a maximum when $XR+YR_1$ is a minimum, since NE is constant.

Now the product $(XR)\times(YR_1)=NRR_1$ is a constant, and hence the sum $XR+YR_1$ is a minimum when $XR=YR_1$—that is, when

$$\frac{X}{Y}=\frac{R_1}{R};$$

that is, when

$$\frac{\text{Number of cells in series}}{\text{Number of groups in parallel}}=\frac{\text{external resistance}}{\text{internal resistance of one cell}}.$$

Since X and Y must be whole numbers it will not always be possible to obtain exactly the best conditions as defined by this ratio.

An important and similar problem is that of an electrical generator of given internal resistance and given e. m. f. It is required to find what the external resistance must be in order to get the maximum power in the external resistance.

Let E be the e. m. f. and R the internal resistance of the generator, and let R_1 be the required external resistance.

The power supplied to R_1 is—

$$I^2R_1,$$

and this is to be a maximum.

But $$I=\frac{E}{R+R_1}.$$

Hence power supplied to R_1 is—

$$\frac{E^2R_1}{(R+R_1)^2}.$$

For a maximum—

$$\frac{d}{dR_1}\left[\frac{R_1}{(R+R_1)^2}\right]=0;$$

i.e.— $$\frac{1}{(R+R_1)^2}=\frac{2R_1}{(R+R_1)^3};$$

or $$R+R_1=2R_1;$$

or $$R=R_1.$$

That is to say, the resistance of the external circuit must be equal to the internal resistance of the generator. In Fig. 180 values of the power supplied to R_1—viz., $\frac{E^2R_1}{(R+R_1)^2}$, are plotted as a function of $\frac{R_1}{R}$, for $E=100$ volts: $R=10$ ohms.

The power in the external resistance is seen to reach a maximum value when $\frac{R_1}{R}=1$.

153. Heating and Cooling Curves.—Assume heat is supplied to a homogeneous conductor at a constant rate of Q calories per second. That is, if a conductor of resistance R ohms is heated by carrying a current of I ampères,

$$Q=0{\cdot}239I^2R \text{ calories per second (§ 145).}$$

Let c be the specific heat of the material of the conductor—*i.e.*, the number of calories to raise the temperature of 1 gm. of the material of the conductor by 1° C.

Let M be the mass of the conductor in grammes.

Let S be the superficial area of the conductor in square centimetres.

Let λ be the cooling constant—*i.e.*, the number of calories lost by the conductor to the surrounding medium in one second per sq. cm. of surface and per 1° C. of temperature difference between the conductor and surrounding medium. [N.B.—The constancy of λ is "Newton's Law of Cooling."]

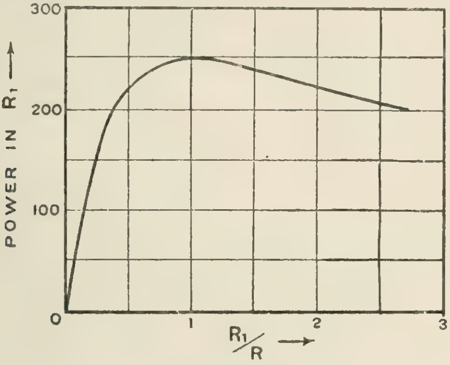

FIG. 180.

In the time element δt seconds the heat supplied to the conductor is $Q\delta t$ calories.

Let θ be the difference of temperature between the conductor and the surrounding medium, and $\delta\theta$ the rise of temperature in the time δt.

The heat supplied, $Q\delta t$, is partly expended in raising the temperature of the conductor by an amount $\delta\theta$, and the remainder is transferred to the surrounding medium.

Hence $$Q\delta t=cM\delta\theta+S\lambda\theta\delta t.$$

Eventually a condition of equilibrium is reached in which the heat supplied to the conductor just balances the heat transferred to the surrounding medium, and the temperature becomes constant.

Let Θ be this steady temperature difference between the body and the surrounding medium.

Then
$$Q = S\lambda\Theta;$$
or
$$S\lambda = \frac{Q}{\Theta}.$$

Therefore
$$Q\delta t = cM\delta\theta + \frac{Q\theta}{\Theta}\,\delta t;$$
or
$$Q\left(1 - \frac{\theta}{\Theta}\right)\delta t = cM\delta\theta.$$

That is,
$$\frac{\delta\theta}{\Theta - \theta} = \frac{Q\delta t}{\Theta cM} = \frac{\delta t}{\tau},$$
where $\tau = \frac{\Theta cM}{Q}$ and is the time which would be required to produce the final temperature rise Θ if no heat were transferred to the surrounding medium.

Since
$$\tau = \frac{\Theta cM}{Q} \text{ and } Q = S\lambda\Theta,$$
$$\tau = \frac{cM}{S\lambda},$$
and is a constant depending on the size, shape, and material of the conductor, but is independent of the rate at which heat is supplied.

The factor τ is termed the time constant of the conductor.

Since
$$\frac{\delta\theta}{\Theta - \theta} = \frac{\delta t}{\tau},$$
$$\int_{\theta=o}^{\theta=\theta} \frac{d\theta}{\Theta - \theta} = \int_{t=o}^{t=t} \frac{dt}{\tau},$$
the initial conditions being $t = o$, and the temperature difference between the body and surrounding medium then being $\theta = o$.

Hence
$$\frac{\Theta - \theta}{\Theta} = e^{-\frac{t}{\tau}};$$
or
$$\theta = \Theta\left(1 - e^{-\frac{t}{\tau}}\right).$$

This equation gives the *temperature rise* θ above the surrounding medium at any time t in terms of the final temperature rise Θ and the time constant τ.

In Fig. 181 a temperature rise curve is shown (Curve I.) as calculated from this equation and for the condition that the time constant τ is equal to 1200 seconds, and the final temperature rise Θ is 70° C.—*i.e.*, an initial temperature of 15° C. is assumed and a final temperature of 85° C.

The equation for the *temperature rise* is thus—

$$\theta = 70\left(1 - e^{-\frac{t}{1200}}\right).$$

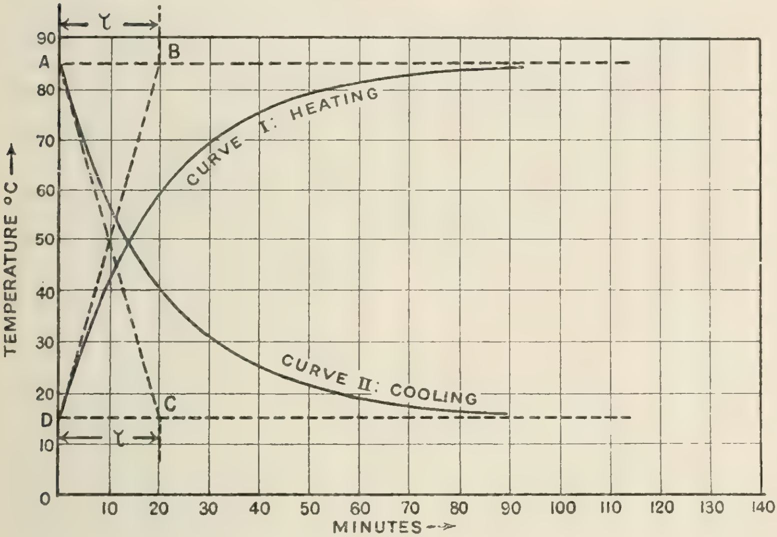

Fig. 181.

Differentiating θ with respect to t in the equation

$$\theta = \Theta\left(1 - e^{-\frac{t}{\tau}}\right)$$

gives—

$$\frac{d\theta}{dt} = \frac{\Theta}{\tau} e^{-\frac{t}{\tau}}.$$

At the time

$$t = o,$$

$$\left.\frac{d\theta}{dt}\right]_{t=o} = \frac{\Theta}{\tau};$$

or

$$\tau = \frac{\Theta}{\left.\frac{d\theta}{dt}\right]_{t=o}}.$$

Hence the time constant is the final temperature rise Θ divided by the initial slope of the curve. Thus, in Fig. 181 the time constant is given by the intercept on the horizontal line through A by the tangent DB to the curve at D.

It is easy to show that if a homogeneous body is brought to a temperature Θ° C. above the surrounding medium and then allowed to cool, the cooling curve will be given by—

$$\theta=\Theta e^{-\frac{t}{\tau}}.$$

This curve is shown as Curve II. in Fig. 181, where the initial temperature rise above the surrounding medium is 70° and the time constant $\tau=1200$ as for Curve I.—that is, Curves I. and II. are respectively the heating and cooling curves for the same body.

The time constant τ of the cooling curve is given by the intercept DC of the tangent to the curve at A and the horizontal line through D.

It will be seen from Curve I. that an indefinitely long time is required for the body to rise to its final temperature, and similarly from Curve II. that an indefinitely long time is required for the body to cool to the temperature of the surrounding medium.

For practical purposes, however, it is only of importance to know what time will elapse before the temperature rise (or fall) reaches within, say, x per cent. of the theoretically calculated final rise (or fall), so that the error incurred in assuming x to be negligibly small is of the same order as the experimental error incurred in measuring the temperature.

The time which elapses before the temperature rise is within x per cent. of the calculated final rise may be found as follows:

$$\frac{\Theta-\theta}{\Theta}=\frac{x}{100};$$

that is—

$$e^{-\frac{t}{\tau}}=\frac{x}{100};$$

or

$$t=\tau\log_e\left(\frac{100}{x}\right).$$

If various values be given to x, the corresponding values of t may be calculated thus:

x	10%	5%	2·5%	1%	0·5%
t	$2{\cdot}3\tau$	3τ	$3{\cdot}7\tau$	$4{\cdot}6\tau$	$5{\cdot}3\tau$

Hence, after a time equal to 4·6 × time constant, the temperature rise has reached a value which is only 1 per cent. less than the final calculated value.

The table also holds for the case of a body cooling down, and gives the time which elapses before the body has cooled down to a temperature which is different from the temperature of the surrounding medium by x per cent. of the temperature difference when the body commenced to cool.

From the equation—

$$t = \tau \log_e \frac{100}{x},$$

it follows that the temperature rise of a body reaches to within x per cent. of its final value in a time which is independent of Q, the constant rate at which heat is supplied to the conductor.

The value of τ for a given body may be determined from its cooling curve (see Fig. 181, Curve II.). Further, if any two points on the heating curve be taken, and if $\theta_1 : t_1$ and $\theta_2 : t_2$ refer to these points respectively, then—

$$\theta_2 - \theta_1 = \Theta\left[e^{-\frac{t_1}{\tau}} - e^{-\frac{t_2}{\tau}}\right];$$

or

$$\Theta = \frac{\theta_2 - \theta_1}{e^{-\frac{t_1}{\tau}} - e^{-\frac{t_2}{\tau}}}.$$

Hence $S\lambda = \frac{Q}{\Theta}$ is determined, and, knowing τ and $S\lambda$, all the data necessary are known for determining the heating curve of the body for any given constant rate of heat supply Q.

154. Graphical Method of determining Pressure Drop in Cables.—The pressure drop at any point in a cable due to a given distribution of load on the cable can always be found arithmetically, but in many cases in practice a graphical method gives the quickest and simplest solution of the problem of determining the pressure drop at any point of a cable from which currents are tapped off from various points along the cable. The principle of this method will be clear from a consideration of the following cases:

Case I.—Cable fed with Current at One End.—In Fig. 182 let GN represent a cable which is fed with current at the end G, and from which currents are tapped off at the points L, M, N respectively—viz., I_L ampères at L; I_M ampères at M; I_N ampères at N.

Let the line GN be so drawn that the length GL represents to scale the resistance r_1 of the length of cable between GL. Similarly LM represents to scale the resistance r_2 of the length of cable between the points L and M, and MN the resistance r_3 of the length of cable between M and N.

Draw ab to scale to represent the current I_L, bc to represent I_M, and cx to represent I_N.

Choose any convenient point O and join Oa, Ob, Oc, and Ox. Draw Os perpendicular to ab, and let r be the resistance represented by the length Os on the same scale that GL represents the resistance r_1.

Take any point G_1 on the vertical line through G and draw G_1L_2 parallel to the line O*a*, and cutting the vertical line through L in L_2. Similarly, draw L_2M_2 parallel to O*b*, M_2N_2 parallel to O*c*, and G_1N_1 parallel to O*x*.

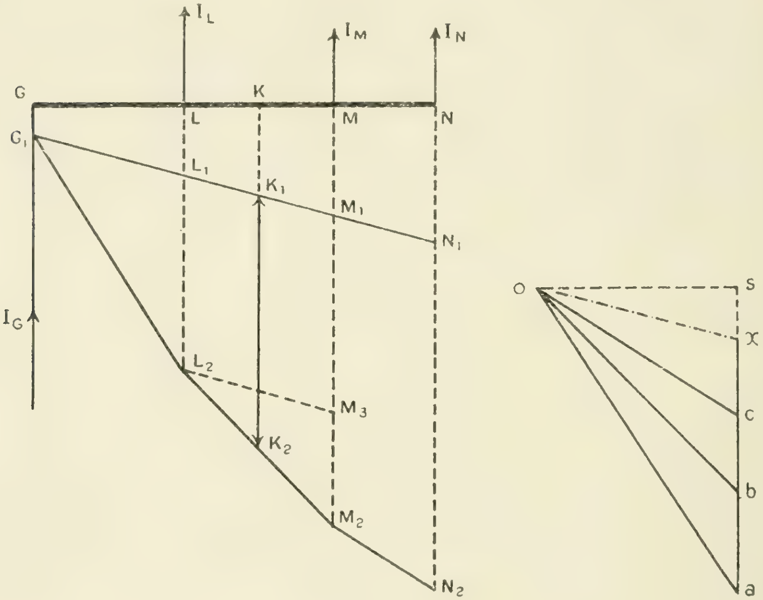

FIG. 182.

The triangles $G_1L_1L_2$ and O*xa* are then similar,

and $$\frac{L_1L_2}{G_1L_1}=\frac{xa}{ox};$$

that is, $$L_1L_2=G_1L_1\,\frac{xa}{ox}\text{ ampères.}$$

But $$\frac{G_1L_1}{GL}=\frac{ox}{os}=\frac{ox}{r}.$$

Therefore $$L_1L_2=GL\,\frac{xa}{r}=(I_L+I_M+I_N)\,\frac{r_1}{r}\text{ ampères.}$$

Hence $L_1L_2\times r$ is the pressure drop along the cable from G to L.

Similarly it may be shown that $M_3M_2\times r$ is the pressure drop along the cable from L to M. That is, $M_1M_2\times r$ is the pressure drop from G to M.

Similarly, $N_1N_2\times r$ is the pressure drop along the cable from G to N.

Hence the pressure drop at any point K on the cable loaded in the manner shown in Fig. 182 is given by—

$$K_1K_2 \times r,$$

where K_1K_2 is the intercept measured on the current scale, between the line G_1N_1 and the line $G_1L_2M_2N_2$ of the vertical through K, and r is a constant resistance given by the length of the line os to the same scale that GL represents the resistance r_1 of the section of the cable between G and L.

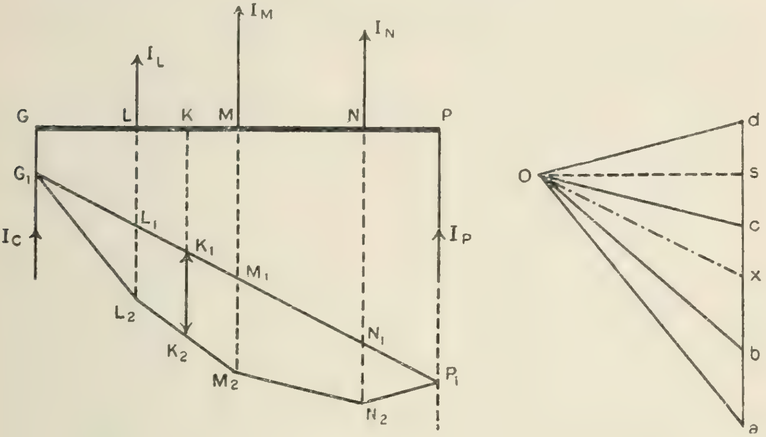

FIG. 183.

Case II.—The cable is fed at each end G and P (Fig. 183), *the points G and P being maintained at the same potential.*

Suppose currents $I_L : I_M : I_N$ ampères are tapped off respectively at the points L, M, and N, and let the length GL represent to scale the resistance r_1 of the portion of the cable between G and L. Similarly LM, MN, NP represent to scale the resistances $r_2 : r_3 : r_4$ of the portions of the cable between L and M, M and N, N and P respectively.

Draw *ab* to represent the current I_L to scale.
,, *bc* ,, ,, I_M ,,
,, *cd* ,, ,, I_N ,,

Choose any convenient point O and join O*a*, O*b*, O*c*, O*d*. Through any point G_1 in the vertical through G draw G_1L_2 parallel to O*a*. Also draw L_2M_2 parallel to O*b*, M_2N_2 parallel to O*c*, and N_2P_1 parallel to O*d*. Join G_1P_1 and draw O*x* parallel to G_1P_1. Then *ax* gives the current fed to the cable at the end G, and *xd* gives the current fed in at the end P. That this is so may be seen from the following considerations:

Consider the point M of the cable. The current I_M tapped off here reaches this point partly from the end P and partly from the end G. Let I_P be the portion of I_M coming from P and I_G the portion of I_M coming from G. If bx is I_G and xc is I_P, then M_1M_2 gives the pressure drop at M due to current flowing in the part GM of the cable. But M_1M_2 will also be the pressure drop at M due to the current flowing in the part PM of the cable. Compare triangle oxd (Fig. 183) with triangle oax (Fig. 182). The construction therefore gives the same pressure drop at M whether considered as due to the current in the part PM or as due to the current in the part GM, and hence the construction given leads to the correct division of current I_M reaching M from the respective ends G and P.

The current I_P is thus given by xd and the current I_G is given by ax.

Draw os perpendicular to ad. Then it can be shown, in a similar manner, as in Case I., that the pressure drop at any point K in the cable is given by—

$$K_1K_2 \times r,$$

where the intercept K_1K_2 is measured on the current scale and r is the resistance corresponding to the length os, on the same scale that GL represents the resistance r_1 of the length of cable between G and L.

Note on Vulcanised Bitumen.—The author is indebted to Messrs. Callender's Cable Co., Ltd., for the following information concerning the electrical properties of vulcanised bitumen. The data refer to two qualities of this insulating material—viz., V.B. No. 1 and V.B. No. 3. The former is hardened by a certain percentage of loading material and the latter is hardened by admixture of a greater percentage and is intended for use in positions where the temperature rise is high or where the cable is subjected to great mechanical stresses.

Material.	*Temperature.*	*Specific Inductive Capacity deduced from Ballistic Measurements.*	*Specific Resistance in Ohms per Cm. Cube.*	*Dielectric Strength in Volts per Cm.*
V.B. No. 1	47° F. 100° F.	4·77 5·74	992×10^{12} 125×10^{12}	79,000
V.B. No. 3	47° F. 100° F.	5·88 7·24	792×10^{12} 140×10^{12}	47,000

CHAPTER XII

ELECTROLYSIS—ELECTRIC ACCUMULATORS

155. Electrolysis.—If two platinum plates be placed at some little distance apart in a liquid conductor (such as acidulated water), and connected to the terminals of a battery, the current decomposes the liquid through which it passes. If the conductor is a liquid metal, however, decomposition of the metal does not take place, the metal being an elementary substance.

This chemical decomposition of a liquid conductor by the passage through it of an electric current is termed **electrolysis,** and the liquid conductor is termed an *electrolyte.* The conductor by which the current *enters* the liquid is termed the *positive electrode* or *anode*, and the conductor by which the current *leaves* the liquid is termed the *negative electrode* or *cathode.*

Liquids which are subject to electrolysis are water rendered conducting; solutions of salts—*i.e.*, compounds of metals and another element or radical (see § 156), salts rendered liquid by fusion, and dilute acids.

The constituents into which a liquid conductor is decomposed by electrolysis do not appear in the mass of the liquid itself, but only at the electrodes.

During electrolysis of a liquid conductor, the metal or hydrogen constituent appears at the *cathode*, and the other element or group of elements—*i.e.*, radical (see § 156)—which was combined with the hydrogen or metal appears at the *anode.*

The constituents into which an electrolyte is separated by electrolysis are called *ions—i.e.*, travellers.

156. Definitions of some Chemical Terms.—The phenomenon of electrolysis is a connecting link between electrical action and chemical action, and the following are some definitions of chemical terms used in connection with electrolytic action.

Radical.—A radical is a group of atoms which persist in chemical changes, behaving as if the group were an element —*e.g.*, the radical sulphion (SO_4).

Valency.—In the case of some elements one atom always combines with one atom of hydrogen; in the case of other elements one atom can combine with two atoms of hydrogen.

If one atom of an element combines with one atom of hydrogen it is said to be *monovalent*—*e.g.*, chlorine—one atom of which combines with one atom of hydrogen to form hydrochloric acid. If one atom of an element combines with two atoms of hydrogen it is said to be *divalent*—*e.g.*, one atom of oxygen combines with two atoms of hydrogen to form water.

Generally stated, the valency of an element (or radical) is the number of atoms of a monovalent element with which that element (or radical) combines.

According to this definition, some elements have more than one valency. For instance, copper in cuprous salts is monovalent, and in cupric salts—*e.g.*, $CuSO_4$—is divalent. A list of valencies of a number of elements is given on p. 224.

Chemical Equivalent.—The chemical equivalent of an element is the weight of that element which is capable of taking the place of, or combining with, one part by weight of hydrogen in a chemical compound.

The chemical equivalent of hydrogen is thus taken as 1.

The atomic weight of oxygen is 16, and the atomic weight of hydrogen is 1·008. Since one atom of oxygen combines with two atoms of hydrogen the chemical equivalent of oxygen is—

$$\frac{16}{2 \times 1{\cdot}008} = 7{\cdot}936.$$

The chemical equivalent is thus equal to the atomic weight divided by the product of the valency and the number 1·008.

Acid.—An acid is a compound of hydrogen and an element or radical. The hydrogen can be replaced by a metal, the resulting residual compound being termed a **salt.**

An acid is monobasic, dibasic, etc., according to the number of replaceable hydrogen atoms it contains. Thus—

Nitric acid, HNO_3, is monobasic.
Sulphuric acid, H_2SO_4, is dibasic.
Phosphoric acid, H_3PO_4, is tribasic.

An **Alkali or Base** is a substance which neutralises an acid with the production of a salt, but with no evolution of hydrogen. A base may be regarded as water in which part of the hydrogen is replaced by a metal, or by a radical which behaves as a metal. Examples of bases are potassium hydrate or caustic potash, KHO, and sodium hydrate, NaHO.

If an acid contains oxygen it is termed an *oxyacid.*

If an acid contains no oxygen it is termed a *hydra-acid*, and its name is formed by the prefix *hydro-*, the element or radical with which the hydrogen in the acid is combined, and the suffix *-ic.* Thus, hydrochloric acid, HCl.

If only one acid compound of an element or radical is known its name ends in *-ic*—*e.g.*, carbonic acid, H_2CO_3. If two be known, the one containing the smaller amount of oxygen takes the suffix *-ous*—*e.g.*, nitrous acid, HNO_2—and the acid containing the larger amount of oxygen takes the suffix *-ic*—*e.g.*, nitric acid, HNO_3.

If more than two acid compounds of an element or radicle be known, the one inferior in oxygen has the prefix *hypo-* and the suffix *-ous*, and the one superior in oxygen has the prefix *per-* and the suffix *-ic*. Thus—

Hypochlorous acid, $HClO$.
Chlorous acid, $HClO_2$.
Chloric acid, $HClO_3$.
Perchloric acid, $HClO_4$.

An acid of which the name ends in *-ous* forms a salt of which the name ends in *-ite*. An oxyacid of which the name ends in *-ic* forms a salt of which the name ends in *-ate*—*e.g.*, mercury when heated with strong sulphuric acid forms mercuric sulphate.

157. Electro-Positive and Electro-Negative Elements.—When binary compounds—that is, compounds of two elements—are decomposed by electrolysis, the two elements make their appearance at opposite poles. Those elements which appear at the *cathode* or negative pole are termed *electro-positive* elements, and those which appear at the *anode* or positive pole are termed *electro-negative* elements.

The difference between the two classes is one of degree only, and they gradually merge into each other.

The following elements are electro-negative to the remaining elements, which are more or less electro-positive: fluorine, chlorine, bromine, iodine, oxygen, sulphur, selenium, tellurium

The tendency of electro-positive elements to unite with electro-positive elements is less than that of electro-positive elements to unite with electro-negative elements.

When binary compounds of oxygen are electrolysed the oxygen invariably appears at the *anode*, since oxygen is electro-negative to all other elements. On the other hand, the hydrogen of hydro-compounds is always disengaged at the *cathode*.

The metals and hydrogen may be arranged in a series according to their power of displacing one another in a salt solution, as follows: potassium, magnesium, aluminium, zinc, cadmium, iron, cobalt, nickel, tin, lead, hydrogen, antimony, bismuth, copper, mercury, silver, platinum, gold.

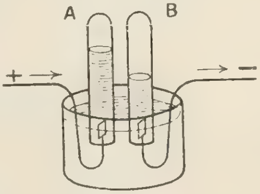

Fig. 184.

158. Electrolysis of Acidulated Water.—Historically, water was the first instance of decomposition by means of the electric current. The apparatus by means of which this electrolysis may be performed is shown in Fig. 184, and is termed a *voltameter*.

Two tubes A and B are filled with acidulated water, and inverted in a vessel which is also partly filled with water. Near the end of each tube is a platinum electrode, and these electrodes are connected respectively to the terminals of a battery. As the current passes in the direction shown, the tubes A and B gradually fill with gas—hydrogen in the cathode tube B, and oxygen in the anode tube A. The volume of hydrogen gas collecting in the tube B is twice the volume of oxygen gas collecting in tube A. These gases are liberated at the respective electrodes, but no trace of the free gases can be observed in the body of the liquid between the electrodes, but are first detected in the immediate neighbourhood of the electrodes.

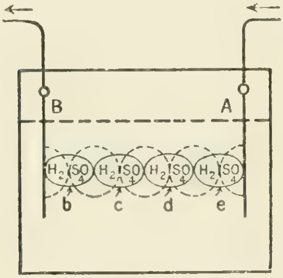

Fig. 185.

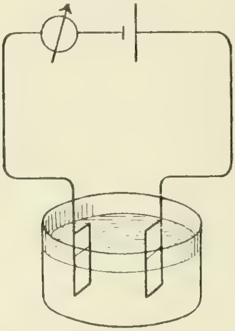

Fig. 186.

The action is assumed to be somewhat as follows: Suppose the water be acidulated with sulphuric acid, H_2SO_4. In Fig. 185 a chain of molecules of H_2SO_4 is shown extending between the electrodes. Consider a molecule such as *d*. It is assumed that the electric force produces a directive action on the ions—*i.e.*, hydrogen H and sulphion SO_4—so that the hydrogen moves with the current, and the sulphion in the opposite direction to the current. The hydrogen of molecule *d* combines with the sulphion of molecule *c*, and the hydrogen of molecule *c* with the sulphion of molecule *b*, so that eventually hydrogen is released at the cathode B. Similarly, the sulphion of molecule *d* combines with the hydrogen of molecule *e* and so on, until eventually sulphion is released at the anode A. The radical sulphion, however, cannot exist in the free state, and consequently attacks the water at the anode, combining with the hydrogen and releasing oxygen. The oxygen and hydrogen thus appear at the respective electrodes as though the water alone were decomposed by the action of the electric current.

159. Polarisation.—If an electric cell—*e.g.*, Daniell's cell (see § 137)—be connected in series with a galvanometer and two platinum electrodes dipping into acidulated water (Fig. 186) no visible chemical decomposition ensues. At first a considerable current is indicated by the galvanometer, but the deflection soon diminishes, and finally becomes very small. If instead of using a single Daniell's cell a source is used, so that the p. d. applied to the platinum electrodes can be varied from some value greater than about 1·7 volts, it is found that when the p. d. is greater than about 1·7 volts a current of considerable strength will flow through the circuit, which after the initial period assumes a steady condition, and is accompanied by chemical decomposition of the acidulated water.

Now disconnect the platinum plates from the electric source and connect them to the terminals of a galvanometer. A current will flow through the galvanometer, and the direction of the current through the acidulated water in which the platinum plates dip will be in the reverse direction to that in which the current flowed when the electric source was connected in the circuit.

The arrangement of the platinum plates in acidulated water through which an electric current is passed from a source of electricity forms an **electrical accumulator**—that is, it stores electricity—and can be used as a secondary source, yielding a current when the two platinum plates are joined by a conductor.

The explanation of this effect is that the passage of an electric current through the acidulated water results in a film of hydrogen being formed at the plate at which the current leaves, and a film of oxygen at the plate at which the current enters. A film of gas is thus formed on each plate, the gas being charged in some such way as was explained in § 135, Chapter X. A p. d. is thus established between the surface of each plate and the liquid, the p. d. between each plate and liquid being different in each case, so that there is a resultant p. d. between the two plates. The direction of this p. d. is opposite to that applied by the external source. In order to actually release the constituents of the water at the electrodes, experiment shows that a p. d. of about 1·7 volts is necessary (see also § 162).

160. Laws of Electrolysis.—Suppose in Fig. 187, A, B, and C represent three voltameters as shown in Fig. 184, and suppose A and B are connected in parallel and the parallel combination in series with C. If a current be passed through the circuit, it is found that the amount of chemical action in C is equal to the sum of the actions in A and B. That is to say, the amount of oxygen liberated in C is equal to the sum of the amounts of oxygen liberated in A and B, and similarly the amount of hydrogen liberated in C is equal to the sum of the amounts of hydrogen liberated in A and B.

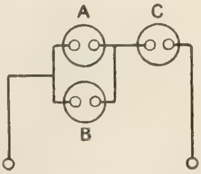

FIG. 187.

Further, the amount of chemical action is found to be independent of the size of the electrodes of the cell, and is directly proportional to the time for which the current flows. In other words, *the chemical action, or mass of substance liberated from an electrolyte, is proportional to the quantity of electricity which passes through the electrolyte.* This is Faraday's first law of electrolysis.

Next suppose a current be passed through dilute sulphuric acid between platinum electrodes, and through a solution of copper sulphate between copper electrodes, these two electrolytic cells being connected in series.

It will be found that the mass of hydrogen liberated in the first cell is to the mass of copper deposited in the second cell as 1·008 : 31·8. In other words, when 1·008 mgs. of hydrogen have been liberated at the cathode in the first cell, 31·8 mgs. of copper will have been deposited at the cathode of the second cell. But these numbers are in the same ratio as the chemical equivalents of hydrogen and copper respectively (see p. 224).

Further, when 1·008 mgs. of hydrogen have been liberated at the cathode of the first cell, 8 mgs. of oxygen will have been released at the anode, and these numbers are also in the same ratio as the chemical equivalent of hydrogen and oxygen respectively.

Faraday's second law of electrolysis states that the *mass of substance liberated from any electrolyte when a given quantity of electricity passes through the electrolyte is proportional to the chemical equivalent of that substance.*

161. Electro-Chemical Equivalents.—In order to liberate 1·008 gms. of hydrogen and 8 gms. of oxygen by the electrolysis of acidulated water, the passage of 96540 coulombs is necessary—that is to say, 1 coulomb liberates $1{\cdot}044 \times 10^{-5}$ gms. of hydrogen.

> **The mass of a substance in grammes liberated by the electrolysis of a liquid when 1 coulomb passes is termed the electro-chemical equivalent of the substance.**

From the results given in § 160 it is known that the electrochemical equivalent is proportional to the chemical equivalent. For hydrogen the chemical equivalent is 1, and hence the constant by which the chemical equivalent of a substance must be multiplied in order to obtain the electro-chemical equivalent of that substance is—

$$=1{\cdot}044 \times 10^{-5}.$$

A list of chemical equivalents and electro-chemical equivalents is given on p. 224.

The amount of a substance liberated by electrolysis in a given time is thus a measure of the current which passes through the electrolyte. So many causes, however, tend to affect the perfect accumulation of the ions at the electrodes that the accurate determination of the electro-chemical equivalent, or conversely to measure the current, in this way is not easy. Large copper electrodes in copper sulphate give good results, but the most satisfactory arrangement is that used by Lord Rayleigh to find the electro-chemical equivalent of silver.

A platinum bowl is filled with a neutral solution of silver nitrate, containing about 15 parts of salt to 100 parts of water. A silver plate wrapped in filter paper dips in the solution, and the current passed from silver plate as anode to platinum bowl as cathode, the current density being about 0·01 ampère per sq. cm.

The bowl is weighed before and after the passage of the current, and the weight of silver deposited is thus obtained. The mean result of the best experiments shows that a current of 1 ampère passed for 1 second—*i.e.*, when 1 coulomb of quantity is passed the mass of silver deposited is 0·001118 gm. This determination is so accurate and convenient that it is used as the practical definition of the ampère (see § 132).

That is to say, *a current of* 1 *ampère is such that, when flowing through a solution of silver nitrate, it deposits silver at the rate of* 1·118 *mgs. per second.*

The chemical equivalent of silver is 107·07, and hence the ratio

$$\frac{\text{electro-chemical equivalent}}{\text{chemical equivalent}}=\frac{0{\cdot}001118}{107{\cdot}07}=1{\cdot}044\times10^{-5},$$

and this number is the same for all ions.

The table given on p. 224 contains some useful data of the commoner elements. The electro-chemical equivalent in each case is obtained by multiplying the chemical equivalent by $1{\cdot}044\times10^{-5}$.

The atomic weight of oxygen is taken at 16, which gives 1·008 for the atomic weight of hydrogen.

162. Energy Involved in Electrolytic Action.—Consider the electrolysis of acidulated water, as described in § 158. From the table on p. 224 it is seen that 1 coulomb of electricity releases $1{\cdot}044\times10^{-5}$ gms. of hydrogen.

If a current of i ampères flows for t seconds, the amount of hydrogen released will be—

$$it\ (1{\cdot}044\times10^{-5})\text{ gms.}$$

Now experiment shows that when 1 gm. of hydrogen combines with oxygen to form water, 34000 units of heat (calories) are released—that is, $34000\times4{\cdot}18\times10^{7}$ ergs $=14{\cdot}2\times10^{11}$ ergs. Hence, to release a mass of $it\ (1{\cdot}044\times10^{-5})$ gms. from water requires an expenditure of—

$$it\times1{\cdot}044\times10^{-5}\times14{\cdot}2\times10^{11}\text{ ergs};$$

that is—

$$1{\cdot}485it\text{ joules (see § 145).}$$

Element.	*Symbol.*	*Atomic Weight.*	*Valency.*	*Chemical Equivalent.*	*Electro-Chemical Equivalent (i.e., Gms. per Coulomb).*
Aluminium.. ..	Al	27·1	3	8·96	0·000093
Bromine	Br	79·96	1	79·32	0·000828
Chlorine	Cl	35·45	1	35·17	0·000367
Copper (from cuprous salts)	Cu	63·6	1	63·09	0·000659
Copper (from cupric salts)	Cu	63·6	2	31·54	0·000329
Gold	Au	197·2	3	65·21	0·000681
Hydrogen	H	1·008	1	1	0·00001044
Iron (from ferrous salts)	Fe	55·9	2	27·73	0·000289
Iron (from ferric salts)	Fe	55·9	3	18·485	0·000193
Lead	Pb	206·9	2	102·63	0·0010714
Mercury (from mercurous salts) ..	Hg	200	1	198·4	0·002071
Mercury (from mercuric salts) ..	Hg	200	2	99·2	0·001036
Nickel	Ni	58·7	2	29·12	0·000304
Nitrogen	N	14·04	3	4·64	0·0000484
Oxygen	O	16·00	2	7·94	0·0000829
Potassium	K	39·15	1	38·84	0·000405
Silver	Ag	107·93	1	107·07	0·001118
Sodium	Na	23·05	1	22·86	0·0002386
Tin (from stannous salts)	Sn	119	2	59·02	0·000616
Tin (from stannic salts)	Sn	119	4	29·51	0·000308
Zinc	Zn	65·4	2	32·44	0·000338

If v is the p. d. at the electrodes during the electrolysis, the energy supplied by the current in sending $i \times t$ coulombs through the electrolyte is vit joules. Hence the applied p. d. must be at least about 1·5 volts, and for this reason one Daniell cell is not sufficient to decompose water (see § 159).

163. Electrolysis and the Electron or Atom of Electricity.—The facts of electrolysis have led to the conclusion that all monovalent atoms carry the same charge of electricity, which is regarded as the ultimate unit of electrical quantity, and is called the **electron.** For each atom of a substance released with its charge at the electrode the corresponding quantity of electricity flows through the electrolyte. For example, if N be the number of atoms in a mass of 1 gm. of hydrogen, and e the electric charge associated with each atom, then—

$$Ne = 96540 \text{ coulombs},$$

since 96540 coulombs are required to liberate 1 gm. of hydrogen

(§ 161), since this is the quantity of electricity which must flow through the electrolyte (acidulated water in this case) to release 1 gm. of hydrogen. Now N has been experimentally determined to be equal to about $60 \cdot 6 \times 10^{22}$ at 0° C., and at a barometric pressure of 760 mm.

$$\text{Hence} \qquad e = \frac{96540}{60 \cdot 6 \times 10^{22}} \text{ coulomb}$$
$$= 1 \cdot 59 \times 10^{-19} \text{ coulomb}$$
$$= 4 \cdot 77 \times 10^{-10} \text{ electrostatic c.g.s. units}$$

(§ 8), and this is the magnitude of the charge of an electron.

If the substance has a valency greater than unity, the atom of the substance is charged with the same number of electrons as the number representing the valency. Thus, an atom of a divalent element carries a charge of two electrons, and so on.

164. Electrolysis and Ohm's Law.—As was stated in § 140, Ohm's Law applies as accurately to electrolytes as to metallic conductors, and the suggestion occurs that the process of conduction is the same in both cases. The view is held that an electric current in an electrolyte is intermittent—that is, a succession of distinct discharges at extremely short intervals. Maxwell has shown that a rapid intermittent charging and discharging can give rise to all the effects of a true resistance. Thus, suppose a condenser of capacity C is alternately charged and discharged at the rate of f times per second by being alternately connected to a battery and short-circuited. If V is the p. d. of the battery, $Q = VC$ is the quantity passing at each discharge, or VCf is the quantity passing per second, and this is the current I.

$$\text{Hence} \qquad I = VCf,$$

and $\frac{1}{Cf}$ represents the resistance, or Cf the conductivity. The greater the value of f, the greater the conductivity, and hence if the electric current is of the nature of an intermolecular discharge, the greater closeness of molecules being probably associated with more rapid charge and discharge corresponds to less resistance—this would explain the greater resistance of electrolytes as compared with metallic conductors as shown in the table, p. 194 (see J. J. Thomson, "Application of Dynamics to Physics and Chemistry").

165. Electrolysis in Practice.—Electrolytic action has most important industrial applications, such as the recovery of metals from their ores by electrolysis, notably aluminium; the refining of copper for use as electrical conductors; electroplating; electro-typing; the isolation of hydrogen and oxygen from water.

An electrolytic meter for measuring the quantity of electricity supplied to a consumer is manufactured by the Reason Company, of Brighton.

Electrolysis is a serious source of corrosion in gas and water pipes bedded in the earth. For example, when such pipes are laid parallel and near to tramway lines which form the return circuit for the tramway supply, the current leaks from the tram lines and flows into and along the pipes for some distance, eventually leaving the pipes to return to the negative main of the supply. At the place where the current leaves the pipes corrosive action is developed during electrolysis of the liquid in the neighbouring earth, and unless special precautions are taken the pipes may in time be completely destroyed.

When iron or steel is immersed in water (especially salt water) it quickly corrodes, and the corrosion is attributed to electrolytic action taking place between different parts of the iron which are not quite in the same physical state—*e.g.*, in boilers in which certain parts have become hardened by hammering, caulking, etc.

The corrosion is specially great when two of the immersed parts of the same structure are of different metals.

In order to prevent this corrosion it is frequently the practice to provide zinc plates immersed in the water and in good electrical contact with the iron or steel plates it is desired to protect. The result of this arrangement is that the corrosion is transferred from the iron to the zinc, and in order to maintain effective protection the zinc plates must be frequently renewed.

A different method of protection has been proposed by E. Cumberland,* although the extent to which this process attains its object has been frequently questioned.

The arrangement comprises a low-tension dynamo generating continuous current at about 6 to 10 volts, and pieces of iron suspended in the water and insulated from the iron plates it is desired to protect—*e.g.*, the plates of a marine boiler.

The boiler plates are connected to the negative pole of the dynamo and the iron pieces suspended in the water connected to the positive pole. Current then flows from the iron pieces to the boiler plates. The suspended iron gradually corrodes, and can be easily and cheaply renewed.

166. Electric Accumulators or Secondary Cells.—It was explained in § 159 that if a current be sent through acidulated water between platinum electrodes a counter e. m. f. is developed at the electrodes, and if the battery be removed, and the electrodes connected by a wire, a current will flow through the electrolytic cell for a short time, the direction of the current being the reverse of that which flowed when the battery was connected in circuit. Such an arrangement is a very simple form of **electric accumulator** or **secondary cell,** although of no practical value in such a simple form, since the storage capacity is extremely small.

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* *Journal of the Inst. of Metals,* 1915, vol. xv.

The earliest form of practical accumulator was Grove's "Gas Battery" invented in 1842. One cell of such a battery is shown in Fig. 188. Two glass tubes, each closed at one end, are filled with, and dip into, acidulated water. Each tube is sealed at the upper end, and contains a platinum plate joined to a platinum wire, which is connected to a terminal outside the tube. Upon passing a current through this cell hydrogen gas collects in the tube from which the current leaves, and oxygen gas in the tube at which the current enters, the volume of hydrogen gas being twice that of the oxygen (see § 158). If the battery now be removed and the two terminals of the Grove's cell joined together, a current will flow from the tube containing the oxygen through the outer connecting conductor to the tube containing hydrogen, and so through the cell. That is to say, the direction of the current through the cell will now be the reverse of that supplied to the cell by the charging battery. The current will continue to flow until the hydrogen and oxygen in the two tubes have completely disappeared.

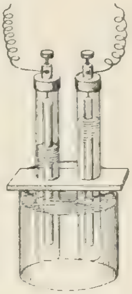


FIG. 188.

167. Lead Accumulators.—In 1860 Planté invented the first lead cell, which consisted of two lead plates immersed in dilute sulphuric acid. When the cell was charged by connecting the plates to the poles of a battery, of e. m. f. sufficient to send a current through the cell, lead peroxide (PbO_2) formed at the anode—*i.e.*, the lead plate connected to the positive pole of the battery—but the lead plate forming the cathode remained chemically unchanged. When the cell was discharged, each of the lead plates became coated with lead sulphate ($PbSO_4$). When again charged, the $PbSO_4$ on the positive plate became changed to lead peroxide (PbO_2) and the $PbSO_4$ on the negative plate became reduced to spongy lead.

The plate which is connected to the positive pole of the battery—*i.e.*, the plate at which the charging current enters the cell—is the positive plate when the cell is discharging.

Planté discovered that by charging and discharging, and then charging in the reverse direction, the capacity of the cell could be greatly increased. By repeating this process many times the cell attained its maximum capacity, and this process is known as "forming" the plates. The surface of the plates is thus rendered more porous, and the active or effective surface enormously increased. The mechanical strength of the plate is, however, greatly reduced by the forming process. In modern cells of the Planté type the plates are often formed by chemical means—*e.g.*, boiling in dilute nitric acid—or electrolytically by adding sodium nitrate to the electrolyte. Care must be taken, however, to remove traces of chemicals, otherwise local action will be set up, and the plates deteriorate.

In 1880 Faure introduced the form of lead cell in which the active material was applied to the plates as a paste, instead of being produced by "forming" the plates. Faure used a paste of red lead (Pb_3O_4) and sulphuric acid for both plates. The mixture formed lead sulphate ($PbSO_4$), and this on the positive plate was converted to PbO_2 by the liberated oxygen during the first charge. The lead sulphate on the negative plate was reduced to spongy lead during the first charge by the combination of the liberated hydrogen with the radicle, SO_4. In the modern construction of the Faure type of plates a grid supports the paste. For the negative plate a paste of litharge (PbO) and sulphuric acid is used, and for the positive plate a paste of red lead and sulphuric acid. During the first charge, the red lead of the positive plate is oxidised to lead peroxide, and the litharge of the negative plate is reduced to spongy lead.

168. Chemical Changes during Charge and Discharge of a Lead Accumulator.—When a secondary cell is charged, the active material of the positive plate is lead peroxide (PbO_2) and of the negative plate is spongy lead. The electrolyte of the cell is dilute sulphuric acid. During *discharge* the chemical change at the positive plate may be represented by the equation—

$$2PbO_2 + 2H_2SO_4 + 2H_2 = 2PbSO_4 + 4H_2O,$$

and at the negative plate by the equation—

$$2Pb + 2H_2SO_4 + O_2 = 2PbSO_4 + 2H_2O.$$

As the cell becomes discharged the density of the electrolyte becomes reduced, and each plate becomes coated with lead sulphate. When charged, the positive plate assumes a rich dark brown and the negative plate a metallic lustre. When discharged, the positive plate becomes a lighter brown and the negative a whitish-grey colour.

The reactions above are only approximate, and the complete reaction is not yet fully understood.

When a cell is fully charged, if the charging current is kept flowing, the hydrogen released at the negative plates and oxygen at the positive plate are evolved as bubbles, and the cell is said to be "gassing." The spray produced by these bubbles bursting on the surface attacks metals (except lead) in the neighbourhood, and hence such metals must be protected by painting with antisulphuric enamel or coating with vaseline.

If a cell remains for some time inactive, or is left uncharged for even a short time, or is discharged below a minimum value of the p. d., a non-conducting layer of white sulphate of lead is formed on the plates, which is very difficult to remove, and resists electrolytic action. When this white sulphate appears, the cell is technically known as "sulphated."

169. Electrical and Physical Changes of the Electrolyte during Charge and Discharge of a Lead Accumulator.—When a cell is fully charged the specific gravity of the electrolyte should be about 1·21, and when discharged about 1·175, the exact values varying slightly with different types of cell. If the specific gravity becomes too high, pure distilled water should be added.

The p. d. at the cell terminals should not be allowed to fall below about 1·8 volts, and when this value of the p. d. is reached the discharge is to be considered complete, and the cell should be recharged immediately. As stated in § 168, if the cell is allowed to remain discharged it rapidly deteriorates and can only be recuperated with difficulty.

When the cell is fully charged the terminal p. d. rises to about 2·7 volts, but immediately current is supplied by the cell the terminal p. d. falls to about 2 volts.

Fig. 189 shows typical charge and discharge curves at constant current which have been supplied by the Tudor Accumulator Company.

Sometimes cells are charged at constant terminal p. d., in which case there is a heavy initial rush of current, and the total duration of the charge is much less than when charging at constant current. Charging at constant p. d. enables the battery to be charged more quickly, but results in a smaller efficiency for the battery.

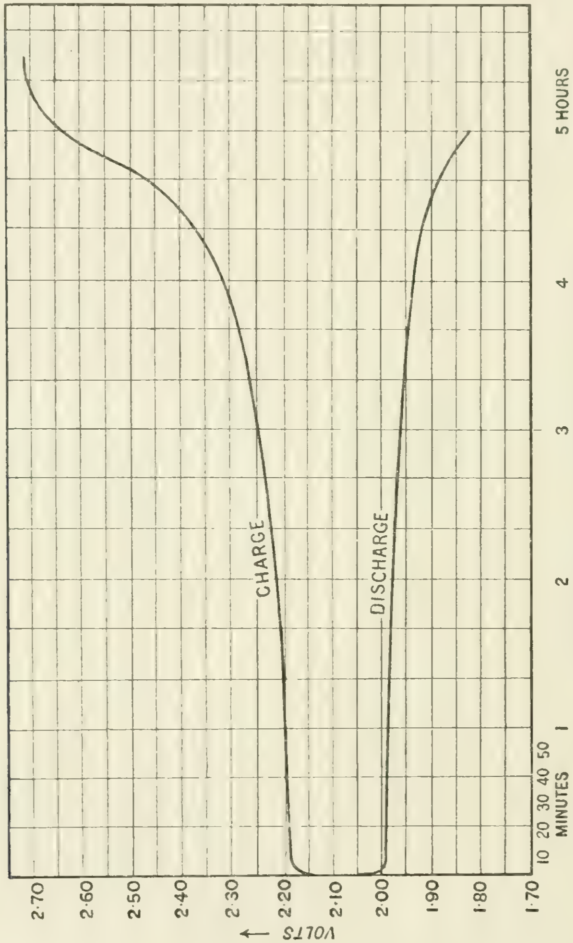
The internal resistance of a cell varies from about 0·002 ohm for a small cell to about 0·00002 ohm for a large cell. The internal resistance of a cell increases as discharge takes place, owing to the formation of water by the chemical action in the cell (see § 168) and consequent diminution of the density of the acid. It is to be observed that the specific resistance of dilute sulphuric acid reaches a minimum when the proportion is about 30 per cent. acid and 70 per cent. water—that is, when the specific gravity is 1·224. Hence when the specific gravity becomes reduced by the formation of water during the discharge of the cell the internal resistance of the cell increases (see § 142).

170. The Capacity of Lead Accumulators.—The ampère-hour capacity of a cell is the number of ampère-hours it can yield at a single discharge—that is, until the terminal p. d. falls to about 1·8 volts. The capacity depends upon the rate of discharge—a high rate of discharge giving a lower capacity than a low rate of discharge.

The capacity of a cell depends upon the temperature at which it is worked—the higher the temperature the greater the capacity. Abnormally high temperatures, however, result in a shortening of the life of the cell.

Overloading a cell reduces the capacity, shortens the life of the cell, and loosens the active material of the plates so that it tends to drop away.

The ampère-hour efficiency of a good type of lead accumulator may be as high as 90 to 91 per cent., but depends upon the rates of charge and discharge respectively.



VOLTAGE CURVES OF TUDOR Q TYPE CELLS WHEN CHARGED AND DISCHARGED AT THE FIVE HOUR RATE.

Fig. 189.

The watt-hour efficiency is much less than the ampère-hour efficiency due to the fact that the p. d. at the terminals of the cell is much higher during charge than during discharge, as shown in Fig. 189. The watt-hour efficiency may be as high as 75 to 77 per cent., but the value in any particular case depends upon the rates of charge and discharge respectively.

The following are data of good types of lead accumulators:

For the Hart Accumulator Company's *traction* batteries (known as "Can-Trac" cells) the capacity varies from about 13·3 watt-hours per pound at the 10-hour rate of discharge to about 11·6 watt-hours per pound at the 5-hour rate of discharge.

For Tudor Accumulator Company's *stationary* batteries, Type LR45, charged at normal rate for 5 to 6 hours at a constant current of 880 ampères:

Time of Complete Discharge.	*Discharge Current.*	*Capacity.*
10 hours	507 ampères	5070 ampère hours
5 hours	845 ampères	4225 ampère hours
3 hours	1211 ampères	3633 ampère hours
1 hour	2535 ampères	2535 ampère hours

It is evident from these figures that the capacity is, for high rates of discharge, relatively small.

The weight of this cell is 2384 pounds, and assuming for the 10-hour rate of discharge that the mean p. d. of the cell terminals during discharge is 1·95 volts, the capacity of this cell is about 4 watt-hours per pound weight of cell.

171. General Remarks on Accumulators.—Secondary cells form a very convenient stand-by in cases of break-down of the main direct current generating plant, and also form a source of supply during periods when it is desired to shut down the main plant—for example, over week-ends. For country house lighting installations the cells may be charged, say, once a week, and then form a very convenient and reliable source of supply for the rest of the week.

For experimental purposes in which a steady current is required the secondary cell is invaluable.

Secondary cells are frequently used in parallel with a dynamo which is required to supply a rapidly fluctuating load. With such an arrangement the dynamo may be designed of sufficient output merely to supply the mean load. When the load exceeds the mean the battery discharges and assists the dynamo, and when the load is less than the mean the battery automatically goes on charge.

In submarine vessels secondary cells are used to drive the boat when running submerged.

The chief disadvantage of secondary cells is their relatively great weight per kilowatt-hour output.

In general, secondary cells are contained in glass boxes for stationary work up to about 1100 ampère-hours' capacity; above this size either lead or lead-lined wood boxes are used. For submarine installations, motor-car work, and general traction purposes ebonite boxes are used on account of their relatively small weight. For ignition batteries for motor-cars celluloid is chiefly used.

172. The Edison Accumulator.—During the last few years this type of accumulator has had considerable practical application. The purpose of the invention was to provide a more robust battery, and one which would be lighter than the lead accumulator for a given capacity, and avoid the use of lead and sulphuric acid and celluloid.

The positive plates have, as active material, nickel hydroxide mixed with fine metallic nickel flake to render it more conducting. Helical tubes, made from perforated nickel steel strips, are filled with alternately arranged thin layers of nickel hydroxide and fine metallic nickel flakes. The tubes are surrounded externally by eight solid steel rings equally spaced for the purpose of internally consolidating the active material. A complete positive plate is an assemblage of such units, a nickel steel frame being used as a mounting for the rows of tubes (see Fig. 190).

The negative plates are an assemblage of small units of oblong pocket form, containing iron oxide incorporated with a trace of mercury to improve its conductivity.

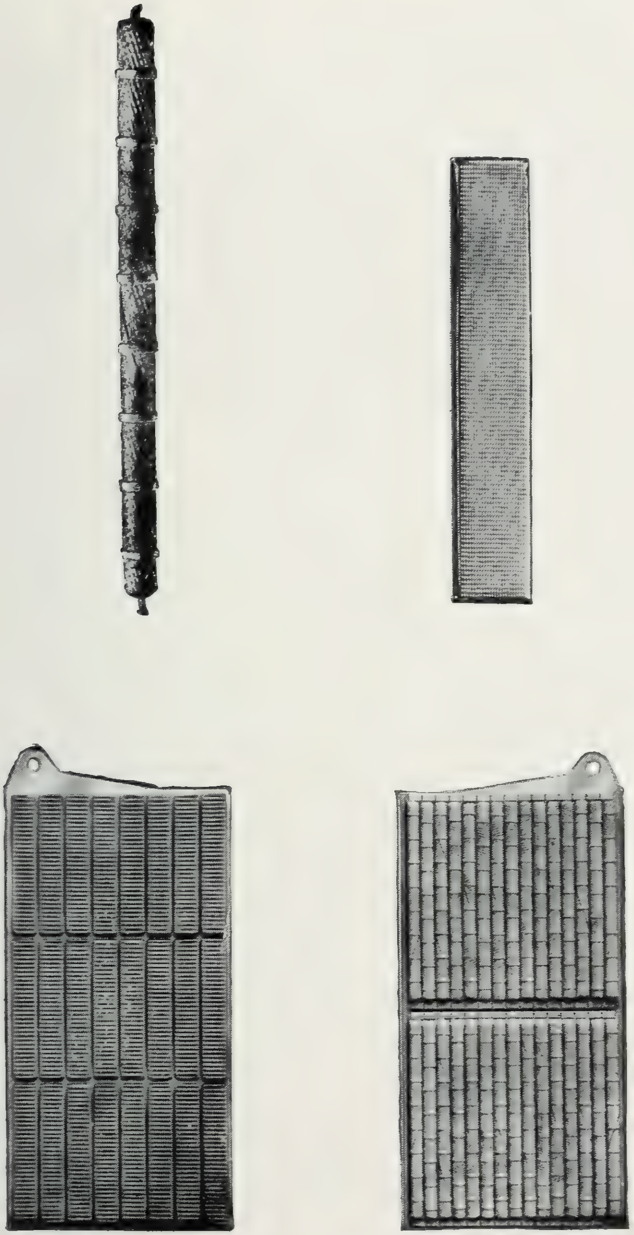
The assembled plates are contained in a nickelled, cold-rolled, sheet-steel box, welded at the seams.

The electrolyte is a solution of potash (KHO) in distilled water, with a small percentage of lithia. The density of the electrolyte does not change to a practically important extent during charge and discharge.

In a new cell nickel hydrate forms the active material of the positive plate, and iron oxide that of the negative plate, and a strong solution of potassium hydrate the electrolyte. On the first charge the green nickel hydrate becomes more highly oxidised, and turns permanently black, and the iron oxide reduced to chemical pure iron. On discharging, the iron plate slowly reoxidises, and the nickel plate becomes reduced, but never again to the lower and original state of oxidisation of green nickel hydrate.

173. Charge and Discharge Curves at Constant Current.—The average working voltage of the Edison cell is 1·2 volts per cell at the normal or 5-hour rate of discharge. During charge a rise of voltage occurs over the normal or 7-hour period of charge. At the beginning of the charge the voltage will be about 1·55 volts, and at the end about 1·84 volts. In Fig. 191 are shown typical charge and discharge curves at constant current.

When charging at constant pressure and completed in normal time of 7 hours, a voltage of 1·67 volts per cell must be maintained throughout the charging period.



To face p. 232.

Fig. 190.

174. Capacity of Edison Accumulators.—The Edison cell gives from 7·5 to 12 ampère-hours per pound of complete assembled battery. At a discharge rate of 1 hour the ampère-hour capacity is 95 per cent. of the ampère-hour output at the normal discharge rate, and this is a feature in which the Edison cell has the advantage over lead cells (see § 170), and is a specially valuable property in connection with electric vehicles, self-starters, and other high-duty service.

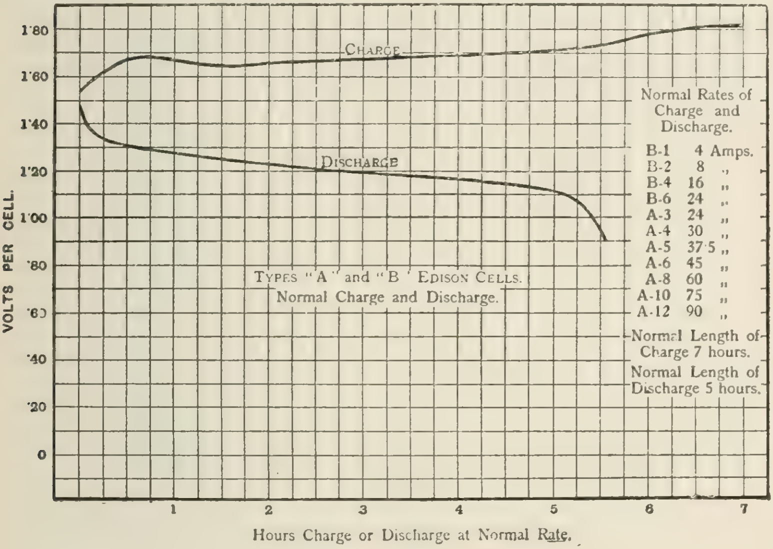


FIG. 191.

The cell gives from 11 watt-hours per pound to about 16 watt-hours per pound, according to the size of the battery.

The ampère-hour efficiency at normal rates of charge and discharge is from 75 to 80 per cent., and the corresponding watt-hour efficiency from 55 to 60 per cent.

175. General Remarks on the Edison Accumulator.—The internal resistance of the 150 ampère-hour capacity cell is about 0·003 ohm when charged. The internal resistance of a cell of capacity K ampère-hours is given by the formula:

$$\text{Internal resistance} = \frac{150}{K} \times 0{\cdot}003 \text{ ohms.}$$

The resistance increases considerably towards the end of the discharge.

A valuable feature of the Edison cell is that it may be allowed to stand on open circuit in any state of charge for long periods, and can stand repeated short circuits and severe overcharging without damage.

The Edison cell behaves in a somewhat remarkable manner at low temperatures. If, after being charged, the temperature of the cell is lowered very considerably, the voltage at the commencement of the discharge will fall off much more than it does normally, but after the discharge has been continued for some time the cells become warmed and resume their normal discharge rate.

For experimental purposes Edison cells, after having been fully charged, have been frozen, and have then given no discharge whatever. On being warmed again, however, their full charge became immediately available.

CHAPTER XIII

THERMO-ELECTRICITY

176. First Principles.—In 1821 Seebeck discovered that in a closed circuit of two metals, if the two junctions of the metals are at different temperatures, an electric current will flow round the circuit. For example, if the metals are iron and copper, soldered together (see § 177), and if one junction is heated a current will flow *from copper to iron across the hot junction* for moderate values of the temperature of the hot junction (Fig. 192). Such currents are termed thermo-electric currents, and two different metals connected together form a thermo-electric couple or junction.

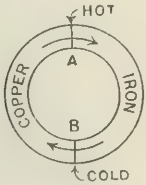


Fig. 192.

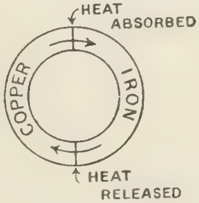


Fig. 193.

In 1834 Peltier showed that when a current flows across the junction of two dissimilar metals it gives rise to an absorption or a liberation of heat. If the current flows in the same direction as the current across the hot junction of a thermo-electric couple, heat is absorbed. If it flows in the same direction as the current cold junction heat is liberated (compare Figs. 192 and 193). This effect is known as the "Peltier effect."

The energy of these thermo-electric currents is to be regarded as being entirely derived from the heat energy supplied, and thermo-electric circuits may be dealt with as perfectly reversible heat engines and the laws of thermo-dynamics applied. In this way important results have been deduced.

The Peltier effect may be accounted for by assuming an abrupt change of potential across the junction of two metals. This p. d. across the junction is not to be confused with the Volta e. m. f. referred to in § 133, the latter being extremely large as compared with the thermo-electric p. d. of a junction.

In 1823 Cumming had shown that there are thermo-electric circuits such that when the temperature of the hot junction is raised above a certain value the thermo-electric e. m. f. in the circuit *decreases.* Thus, suppose that a thermo-electric circuit be made of copper and iron and let the cold junction be kept, say, at a temperature of 20° C. As the temperature of the hot junction is raised the e. m. f. in the circuit increases until the temperature of the hot junction reaches a value of about 260° C., when the e. m. f. has its maximum value. If the temperature of the hot junction is still further increased the e. m. f. decreases, and eventually becomes zero and then reverses in direction. The temperature of the hot junction for which the e. m. f. is a maximum is a definite temperature for a given pair of metals, and does not depend on the temperature of the cold junction. This temperature, which, as already stated, is about 260° C. for an iron-copper couple, is termed the *neutral* temperature. *If the mean temperature of the junctions is the neutral temperature there will be no e. m. f. in the circuit* (see p. 241).

In Fig. 192, suppose the junction A is kept at the neutral temperature 260° C. It follows from what has been said that the e. m. f. round the circuit will be in the same direction whether the temperature of the junction B is below 260° C. or above 260° C., as shown by E in Figs. 194*a* and 194*b*. For if the junction B is below 260° C., the current will be from copper to iron across the *hotter* junction A, whereas if the junction B is above 260° C., the current will be from copper to iron across the *colder* junction A.

Lord Kelvin (Sir W. Thomson) deduced from these results that there must be reversible thermo-electric effects in the circuit other than the Peltier effect, and that an e. m. f. must be produced in an unequally heated conductor and due to the temperature differences along the conductor. This deduction has been verified by direct experiment, and the effect is known as the "Thomson effect."

The Thomson effect corresponds to an e. m. f. directed from a hot point to a cold point in an *iron* conductor, and from a cold point to a hot point in a *copper* conductor, these e. m. f.'s being due to the temperature differences at different points in the conductor. The Thomson effect is said to be positive in copper. In iron below a dull red heat the effect is negative, above dull red and below bright red the effect is positive, and above bright red is again negative. In nickel below a temperature of about 210° C. the effect is negative, and above about 210° C. is positive, and again becomes negative above about 310° C. The curious change of direction of the Thomson effect at certain temperatures is suggestive of some connection with the change of magnetic properties of these metals at the critical temperatures (see § 119).

The Thomson effect in the metal lead is inappreciable, and lead is therefore taken as the standard metal of reference for thermo-electric effects.

In Fig. 194*a* the mean temperature of the two copper-iron junctions is 250° C.—*i.e.*, *less than* the neutral temperature—and consequently the e. m. f. is directed from *copper to iron across the hot junction.*

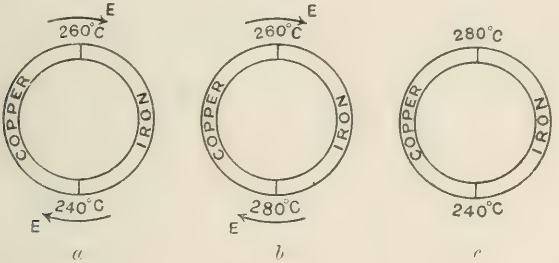


Fig. 194.

In Fig. 194*b* the mean temperature of the two junctions is 270° C.—*i.e.*, *greater than* the neutral temperature—and consequently the e. m. f. is directed from *iron to copper across the hot junction.*

In Fig. 194*c* the mean temperature of the two junctions is 260° C.—*i.e.*, the neutral temperature—and there is therefore no resultant thermo-e. m. f. in the circuit.

177. Laws of Thermo-Electric Circuits.—Experiments have shown that the following two statements are true:

1. *Law of Successive Temperatures.*—Let E be the e. m. f. when the cold junction is at t_1° and the hot junction at t_2°.

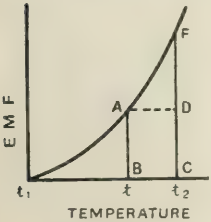


Fig. 195.

Let E_1 be the e. m. f. when the cold junction is at t_1° and the hot junction at t°.

Let E_2 be the e. m. f. when the cold junction is at t° and the hot junction at t_2°.

Then $$E=E_1+E_2.$$

That is to say, if the curve in Fig. 195 shows the e. m. f. as a function of the temperature of the hot junction when the cold junction is kept at the temperature t_1°, then if the hot junction is at t° the e. m. f. is given by $E_1=AB$. If the hot junction is at the temperature t_2° the e. m. f. is given by $E=FC$. If, now, the cold junction is at the temperature t° and the hot junction at t_2° the e. m. f. is—

$$E_2=FD;$$

or $$E_2=E-E_1.$$

2. *Law of Intermediate Metals.*—Consider any thermo-electric circuit, composed of two metals A and B, of which the cold junction is at the temperature t_1° and the hot junction at the temperature t_2° (Fig. 196). The thermo-electric e. m. f. round the circuit is not altered if the hot junction is opened and any number of different metals, such as CD, are inserted (Fig. 197), and the cold junction opened and any number of different metals inserted, provided that each junction AC : CD : DB is at the temperature t_2° and each junction AE : EF : FB at the temperature t_1°.

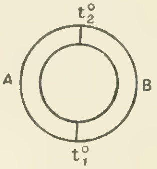


FIG. 196.

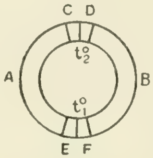


FIG. 197.

The importance of this result is that in forming a thermocouple it is immaterial whether the dissimilar metals are merely twisted together or soldered. Further, if, say, the inserted metals E and F (Fig. 197) correspond to leads of a galvanometer, no effect is produced whatever the metals forming such leads may be, provided the junctions of all such leads be maintained at the same temperature.

178. The E. M. F. of a Thermo-Electric Couple.—Experiment shows that for a wide range of temperature the e. m. f. of many thermo-electric couples may be written as follows:

$$E = a\,(T_2 - T_1) + b\,(T_2^2 - T_1^2) \text{ volts},$$

where a and b are constants, depending on the nature of the metals forming the couples. In this formula T_2 is the absolute temperature of the hot junction—*i.e.*, the temperature in °C. plus 273°—and T_1 the absolute temperature of the cold junction. This expression may be rewritten—

$$\begin{aligned} E &= (T_2 - T_1)\left[a + 2b\,\frac{T_2 + T_1}{2}\right] \\ &= (t_2 - t_1)\,(c + 2bt_m), \end{aligned}$$

where t_2 and t_1 are the temperatures in °C. of the hot and cold junctions respectively, t_m is the mean of t_1 and t_2, and $c = a + 2b \times 273$.

Now suppose $t_1 = 0°$ C.

Then $$E = t_2\,(c + bt_2).$$

Differentiating E with respect to t_2 gives—

$$\frac{dE}{dt_2}=c+2bt_2.$$

This expression gives the increase of e. m. f. per 1° C. increase of temperature of the hot junction. If $\frac{dE}{dt_2}$ is plotted as a function of t_2, a straight line will be obtained, as shown in Fig. 198. As stated, the e. m. f. of the thermo-couple, when the hot junction is at a temperature t_2° C., and the cold junction at a temperature 0° C., is—

$$E=t_2\,(c+bt_2)$$
$$=t_2\times\text{ordinate EF for the mean temperature } \frac{t_2}{2}{}^\circ\text{ C.}$$

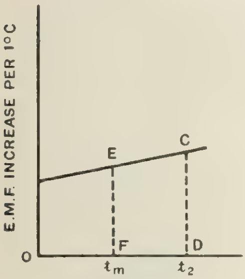


Fig. 198.

Hence the e. m. f. is the area contained between the axes of co-ordinates, the ordinate CD, and the sloping line EC (Fig. 198).

If the cold junction is at a temperature t_1° C., say, OF in Fig. 198, and the hot junction at the temperature t_2° C., the e. m. f. is—

$$E=(t_2-t_1)\left[c+2b\,\frac{(t_2+t_1)}{2}\right].$$

But $c+2b\,\frac{(t_2+t_1)}{2}$ is the value of $\frac{dE}{dt}$ for the mean temperature $\frac{(t_2+t_1)}{2}$, and hence the e. m. f. is given by the area CDEF (Fig. 198).

As was stated in § 176, the metal lead does not show the Thomson effect, and for this reason lead is usually taken as the standard metal of reference, and the thermo-electric e. m. f. of any other metal given as the e. m. f. in a circuit formed by that metal and lead.

In Fig. 199 the values of $c+2bt_2$ are plotted in micro-volts per 1° C. as a function of the temperature t_2 of the hot junction for a number of metals referred to the metal lead as the standard. That is to say, the values of c and $2b$ for lead are taken to be zero.

If the values of $c+2b\,\frac{t_2+t_1}{2}$ for a thermo-couple of any metal and lead lie above the lead line for the mean temperature $\frac{t_2+t_1}{2}$ of the two junctions, the e. m. f. is from that metal to lead across the *cold* junction, and if the line lies below the lead line the e. m. f. is from that metal to lead across the *hot* junction.

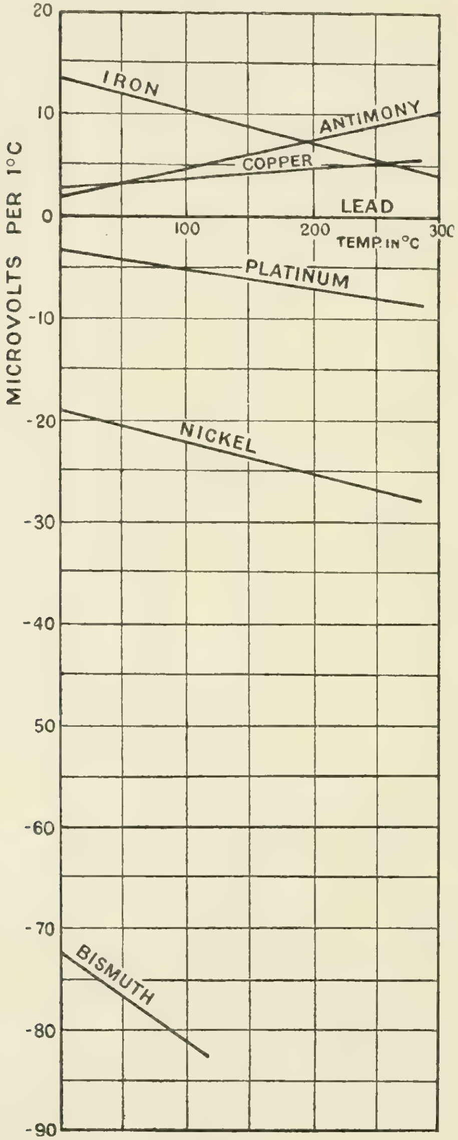


Fig. 199.

Suppose that a thermo-couple circuit be formed of iron and copper, as shown in Fig. 200.

In accordance with Law II., given in § 177, it follows that for a given mean temperature of the junctions the thermocouple circuit of Fig. 200 gives the same total e. m. f. round the circuit as that of Fig. 201, provided that the two junctions A_1A_2 of Fig. 201 are at the same temperature t_2°, and the two junctions B_1B_2 are at the same temperature t_1°.

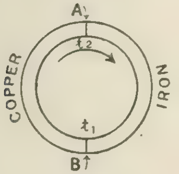


FIG. 200.

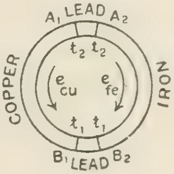


FIG. 201.

Suppose in Fig. 201 the mean temperature of the junctions A_1A_2 and B_1B_2 is—

$$t_m = \frac{t_1 + t_2}{2} = 100° \text{ C.}$$

From Fig. 199 the total e. m. f. across the junctions A_2B_2 from A_2 to B_2 is—

$$e_{fe} = \frac{10 \cdot 2}{10^6} (t_2 - t_1) \text{ volts.}$$

Similarly the total e. m. f. across the junctions A_1B_1 from A_1 to B_1 is—

$$e_{cu} = \frac{3 \cdot 6}{10^6} (t_2 - t_1) \text{ volts.}$$

Hence the total resultant e. m. f. round the circuit is—

$$e = e_{fe} - e_{cu} = \frac{6 \cdot 6}{10^6} (t_2 - t_1),$$

and acts from copper to iron across the *hot* junction A, as shown in Fig. 200.

If $t_2 - t_1 = 100°$ C.

$e = 660$ micro-volts.

If the mean temperature of the couple is about 260° C. the e. m. f. is zero, and if the mean temperature is greater than about 260° C. the direction of the e. m. f. of the couple becomes reversed—that is, acts from copper to iron across the *cold* junction. This reversal of the direction of the e. m. f. is known as *thermo-electric inversion*, and the mean temperature at which it occurs is called the *neutral temperature*, and is the temperature at which the lines in Fig. 199 for the two metals forming the couple intersect

From the foregoing it will be seen that the e. m. f. round a thermo-electric circuit for any given couple and for any given value of the mean temperature of the junctions $t_m = \frac{1}{2}(t_1 + t_2)$ may be obtained as follows:

In Fig. 199 find the magnitude in micro-volts of the intercepted ordinate at the temperature t_m between the lines for the two metals forming the couple, and multiply this by the temperature difference $t_2 - t_1$ between the hot and cold junctions. Thus, for an iron-copper thermo-couple, for $t_2 = 150°$ C. : $t_1 = 50°$ C. : $t_m = 100°$ C. : $t_2 - t_1 = 100°$ C. :

$$\begin{aligned} e &= 6{\cdot}6 \times 100 \text{ micro-volts} \\ &= 660 \text{ micro-volts, as previously found.} \end{aligned}$$

The lines given in Fig. 199 are only to be taken as a general guide for any particular practical case, as the values depend on the purity of the metals. The e. m. f. of any couple used, for example, for temperature measurements is found by actual calibration (see § 179).

179. Practical Application of Thermo-Electric Couples.— Thermo-electric couples are frequently used for the measurement of high temperatures (up to about 1400° C.). The couple may be connected to a high-resistance galvanometer or voltmeter, so that the resistance of the couple itself is negligible, and the reading of the voltmeter is then very approximately proportional to the e. m. f. of the couple. The voltmeter may be calibrated to read temperatures directly for a given temperature of the cold junction. For more accurate measurements the e. m. f. of the couple may be opposed by an e. m. f. of known value, and a galvanometer connected in circuit to give zero deflection when the two opposing e. m. f.'s are equal.

The metals used for thermo-electric temperature measurements depend upon the magnitude of the temperature it is desired to measure. For temperatures below about 500° C., couples of copper-constantan, copper-nickel, or iron-constantan are frequently used. In Fig. 202 is shown the e. m. f. increase per 1° C. temperature difference of the junctions, plotted as a function of the temperature, for thermo-couples of copper-nickel and copper-constantan* respectively. Thus, if the hot and cold junctions of the copper-nickel couple are respectively at temperatures 200° C. and 100° C., the mean temperature is 150° C., and the e. m. f. is given by the area ABCD—*i.e.*, 2700×10^{-6} volts.

Since the line for the copper-nickel couple is nearly parallel to the axis of temperature, the e. m. f. for a given temperature difference $t_2 - t_1$ is roughly equal to $(t_2 - t_1)\ 27 \times 10^{-6}$ volts for all values of t_2 and t_1 over the range for which such a couple may be used (about 0° to 500° C.).

---

* Constantan is 60 per cent. Cu and 40 per cent. Ni.

For the copper-constantan couple, however, the line is steeper, and the e. m. f. developed depends not only on the temperature difference, but also to a great extent on the mean temperature of the two junctions.

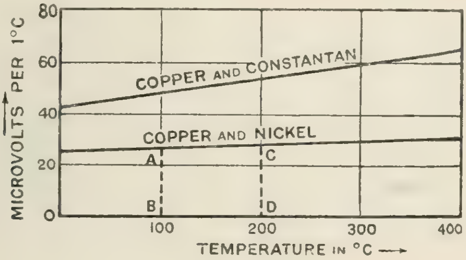


Fig. 202.

The table given on p. 244 refers to a Student's Pyrometer, manufactured by Mr. R. W. Paul. The instrument consists of three thermo-couples—viz., iron-eureka, copper-eureka, and copper-iron. The table gives the details of iron-eureka and copper-eureka couples. For the copper-iron couples the e. m. f. is the difference of the e. m. f.'s for the iron-eureka and copper-eureka couples respectively. The resistances of the respective couples from terminal to terminal are: iron-eureka = 0·95 ohm; copper-eureka = 0·85 ohm: copper-iron = 0·2 ohm.

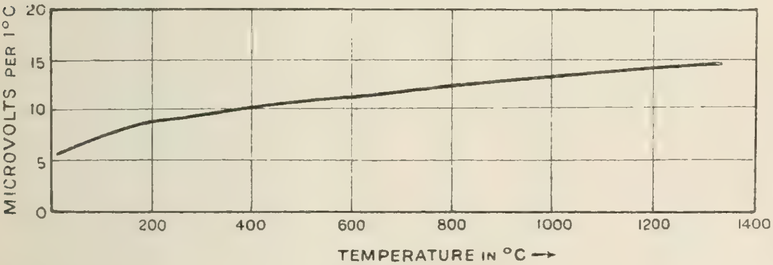


Fig. 203.

It is to be observed that, for the iron-eureka couple, the e. m. f. for 1° C. temperature difference of the junctions is nearly constant over the whole range of temperatures, and the e. m. f. for this couple is very approximately 0·0576 $(t_2 - t_1)$ millivolt, where $t_2 - t_1$ is the temperature difference of the hot and cold junctions in °C.

For temperatures from about 300° C. to 1400° C. thermo-electric couples of platinum and a platinum alloy—*e.g.*, platinum iridium or platinum rhodium—should be used.

In Fig. 203 is given a calibration curve, showing the values of the e. m. f. increase per 1° C. increase of temperature differ-

Temperature. °*C.*	*Open Circuit E. M. F. Cold Junction at 0° C.*			
	Iron Eureka.		*Copper Eureka.*	
	Millivolts	*Millivolts per °C.*	*Millivolts.*	*Millivolts per °C.*
10	0·6	0·058	0·4	0·040
20	1·2	,,	0·8	1
30	1·7	,,	1·2	2
40	2·3	,,	1·6	3
50	2·9	,,	2·1	4
60	3·5	,,	2·5	5
70	4·0	,,	3·0	5
80	4·6	0·057	3·4	6
90	5·2	,,	3·9	7
100	5·8	,,	4·3	7
110	6·3	,,	4·8	8
120	6·9	,,	5·3	9
130	7·5	,,	5·8	9
140	8·0	,,	6·3	0·050
150	8·6	,,	6·8	1
160	9·2	,,	7·3	1
170	9·7	,,	7·8	2
180	10·3	,,	8·3	2
190	10.9	,,	8·8	3
200	11·5	,,	9·4	4
210	12·0	,,	9·9	4
220	12·6	,,	10·4	5
230	13·2	,,	11·0	5
240	13·7	,,	11·5	6
250	14·3	,,	12·1	6
260	14·9	,,	12·7	7
270	15·5	,,	13·2	8
280	16·1	,,	13·8	8
290	16·6	0·058	14·4	9
300	17·2	,,	15·0	9
310	17·8	,,	15·6	0·060
320	18·4	,,	16·2	0
330	18·9	,,	16·8	1
340	19·4	,,	17·4	1
350	20·1	,,	18·0	1
360	20·7	,,	18·6	2
370	21·3	,,	19·2	2
380	21·8	,,	19·8	2
390	22·4	,,	20·5	3
400	23·0	,,	21·1	3
410	23·6	,,	21·7	3
420	24·2	,,	22·3	3
430	24·7	,,	23·0	4
440	25·3	,,	23·6	4
450	25·9	,,	24·3	4
460	26·5	0·058	24·9	4
470	27·1	,,	25·5	4
480	27·6	,,	26·2	4
490	28·2	,,	26·8	4
500	28·8	,,	27·5	—

ence of the hot and cold junctions for a platinum and platinum rhodium thermo-couple supplied by the Cambridge Scientific Instrument Company.

Thermo-couples can be made so that they give a very small time lag of e. m. f. on temperature change, and thus form a valuable means of obtaining a record of rapidly varying temperatures.

Thermo-couples may be used to measure the temperature at a given spot in a non-uniformly heated substance—*e.g.*, temperature at or near the middle of the cross-section of a field coil.

For small differences of temperatures (a maximum temperature of about 80° C.) a bismuth-antimony couple is often used on account of the high thermo-electric e. m. f. of such a couple (see Fig. 199).

Another interesting application of thermo-electric effects is found in the Duddell thermo-galvanometer, a diagrammatic view of which is shown in Fig. 204.

A single loop of silver wire L is suspended by means of a quartz fibre Q between the pole pieces NS of a permanent magnet. The loop is surmounted by a glass stem G carrying a mirror M, whilst its lower ends are connected to a bismuth-antimony couple. A heating resistance or "heater," consisting of a fine filament of high specific resistance, usually a platinised quartz fibre, is fixed immediately under the thermo-couple. When the current to be measured is passed through the heater, part of the heat generated is radiated and carried by convection to the thermo-junction, and the resultant e. m. f. developed produces a current in the loop L, causing it to turn in the magnetic field of NS (see Section IV.). The deflection of the loop L is measured by means of a spot of light reflected from the mirror M to a distant scale.

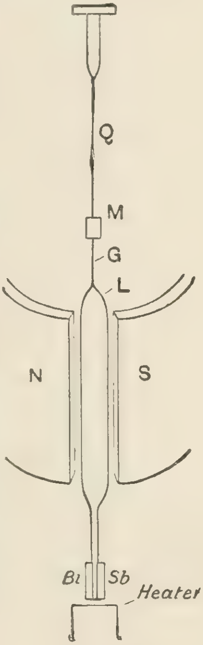


FIG. 204.

The instrument is sufficiently sensitive to measure a current as small as 20×10^{-6} ampères (20 micro-ampères), whilst currents as small as 2·2 micro-ampères may be detected.

In the *Electrical World* for 1914 test data are reproduced, as given by Dr. Kollert of Chemnitz, for a thermo-electric generator, capable of supplying about 1 kilowatt of electrical energy at a terminal pressure of about 30 volts. The figures given are as follows: For a temperature difference of about 313° C. between the hot and cold junctions, each couple developed a pressure of 0·077 volt on open circuit, and this pressure fell to 0·0412 volt when 28·2 ampères were passing through the circuit. The couples were made of an alloy of a *secret composition* and a Cu-Ni alloy, built up in wedge-shaped sets, so that ten of these sets formed a ring which could be clamped round the heating tube, a simple and robust construction being thus obtained. The figures in the following table show the cost of the installation of such a thermo-electric generator as compared with steam, gas, and petroleum driven plants respectively, and also the relative costs of obtaining electrical energy by the different methods:

Type of Generator of Electricity.	*Cost of Installation.*	*Cost of obtaining Electrical Energy.*
Thermo-electric generator	13	5·6
Steam plant	26	24
Gas plant	30	16·5
Petroleum plant	38	19·3

The enormous practical importance, as regards economy of fuel consumption, of such results obtained from a thermoelectric generator, if these results can be confirmed and reproduced, is obvious. It may, therefore, be possible to use thermocouple effects, not merely for measuring instruments as at present, but as an extremely efficient substitute for the motor-driven dynamo.

SECTION IV

ELECTRO-MAGNETISM

CHAPTER XIV

MAGNETIC EFFECTS OF ELECTRIC CURRENTS

180. Electro-Magnetism.—Electro-magnetism deals with the magnetic effects due to electric currents, and comprises the mutual action between magnetic fields and current-carrying circuits.

In 1820 Oersted discovered that a wire carrying an electric current is surrounded by a magnetic field due to the current. Suppose a wire (Fig. 205) is held in the magnetic meridian parallel to a compass needle and a little distance *above* the needle. If an electric current is caused to flow along the wire in the direction from south to north, the compass needle will be deflected so that the N pole moves towards the west. If the current in the wire flows in the direction from north to south the N pole of the needle will be deflected towards the east.

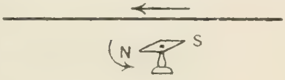


Fig. 205.

If the wire is placed *below* the compass needle and the current is directed from south to north, the needle will be deflected so that the N pole moves towards the east, and if the current flow from north to south the N pole will be deflected towards the west. The following rule gives the connection between the direction of the current in a conductor and the direction of the magnetic lines of force due to the current:

Imagine a right-handed screw with its axis in the conductor and pointing in the direction in which the current is flowing. The direction of the lines of force is the same as that in which the screw must be turned to cause it to advance in the direction of the current.

This rule is illustrated in Fig. 206, and it will be seen that it enables the direction of deflection of the compass needle in Fig. 205 to be predicted.

Suppose now that a wire is bent to a rectangular shape (Fig. 207) and placed with its plane in the magnetic meridian, a compass needle being supported in the plane of the rectangle. If a current be sent round the coil in the direction shown in

Fig. 207 it follows, from what has been said, that the action of the current in both the horizontal sides is to deflect the needle so that the N pole moves towards the west. From a consideration of the rule given above it will be seen that the action of the current in the vertical sides also deflects the N pole towards the west. If the coil is wound with a number of turns and the current in the coil maintained at its original value, each additional turn of the coil increases the deflecting effect on the compass needle, and, as will be seen later, the magnetic action of the current on the needle will be proportional to the product of the magnitude of the current and the number of turns in the coil. If the current is measured in ampères the product of current strength into the number of turns in the coil is termed the *ampère-turns* of the coil. The arrangement shown diagrammatically in Fig. 207 is one form of galvanometer.

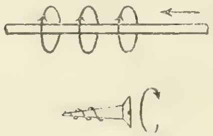


Fig. 206.

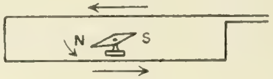


Fig. 207.

181. Ampère's Theorems.—The magnetic effect of a current-carrying circuit was stated by Ampère as follows:

> **The magnetic field produced by a current in a plane closed circuit is, at all points for which the distances from every part of the conductor are great in comparison with every dimension of the circuit, the same as that produced by a small magnet placed anywhere within the circuit with its axis at right angles to the plane of the current, and of which the magnetic moment is proportional to the current flowing and to the area of the circuit.**

The experimental proof of this statement may be made as follows:

Suppose a circular plane coil of wire be taken, the ends leading from the coil being twisted together so that the currents in these leads neutralise each other's magnetic effects.

Let the coil be placed with its plane in the magnetic meridian. Suppose a current be sent through the circuit of the coil and a small compass needle be placed on the east-west line through the centre of the coil. If the distance of the needle from the plane of the coil be great compared with the dimensions of the coil, it is found from observations of the deflection of the compass needle that the force exerted on the needle is in the inverse ratio of the cubes of the distances of the centre of the needle from the plane of the coil. In fact, the force varies in precisely the same way as that due to a small magnet placed in the plane of the coil and with its axis perpendicular to the plane of the coil. The force due to a small magnet at a distant point on the prolonged axis of the magnet was considered in § 99, Section II.

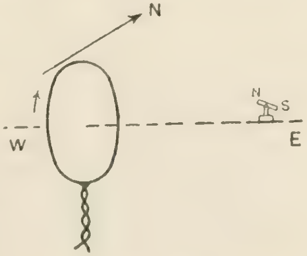


Fig. 208.

Further, it is possible so to adjust the current strength in the coil that the action on the distant compass needle is precisely the same as that of a small magnet of given moment placed in the plane of the coil and with its axis perpendicular to the plane of the coil. If the current strength (as defined in § 145 by the Joule effect) is doubled, the moment of the equivalent magnet is doubled, and if for a given current the area of the coil is doubled, the moment of the equivalent magnet is also doubled.

The general statement of Ampère is thus a statement of experimental facts. This equivalence of a current circuit and a magnet leads to an *electro-magnetic definition of unit current*, as is more fully explained in § 185.

The second theorem of Ampère may be immediately deduced from the foregoing results, and may be stated as follows:

> **Every linear conductor carrying a current is equivalent to a simple magnetic shell, the bounding edge of which coincides with the conductor, the moment per unit area of this equivalent shell—*i.e.*, the strength of the shell—being proportional to the strength of the current.**

This theorem may be proved as follows:

In Fig. 209 let ABC be the circuit in which a current of strength C is flowing in, say, a clockwise direction, as shown. A network of conductors may be constructed as in Fig 210,

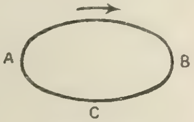


Fig. 209.

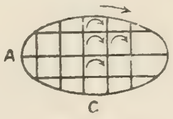


Fig. 210.

each mesh carrying a current of strength C, also flowing in a clockwise direction. It is clear, then, that in every conductor which is common to two meshes equal and opposite currents will flow, and consequently will neutralise each other's effect. Thus the current in the system of meshes is equivalent to the current flowing in the boundary conductor ABC.

Now by the first proposition of Ampère stated above, the current in each mesh may be considered as equivalent to a small magnet of moment proportional to the current in the mesh and to the area of the mesh, or what is the same thing, to a magnetic lamination uniformly magnetised and filling the area of the mesh. Hence the whole network of meshes is equivalent to a magnetic shell of strength proportional to the current, and of which the boundary edge coincides with the conductor carrying the current. The magnetic forces due to a current flowing in a circuit are therefore the same as those due to a magnetic shell of boundary edge coincident with the conductor carrying the current, and the results deduced in §§ 103–105, Section II., are immediately applicable to the case of an electric circuit.

182. Force due to Current in a Long Straight Conductor.—In Fig. 211 let the conductor be perpendicular to the plane of the paper, and let A represent the trace of the conductor. The magnetic effect due to a current in this conductor is the same as that due to a magnetic shell of which the bounding edge is coincident with the conductor. Since the conductor is assumed to be very long, the magnetic shell may be assumed to be bounded by the conductor and to go off to infinity in other directions. Thus the shell may be such that its trace in the plane of the paper is given by ABC, and extending to infinity in the direction AC, and also to infinity in directions perpendicular to the plane of the paper.

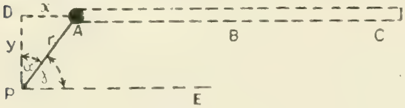


Fig. 211.

If it be assumed that the direction of the current in the conductor A is *away* from the observer, it will be seen, from what was said in § 180 as to the direction of the lines of force due to this equivalent current, that the side of the shell facing P in Fig. 211 will be the positive side.

The potential at the point P due to the shell is by § 103—

$$\Phi\omega,$$

where Φ is the strength of the shell and ω is the solid angle subtended by the shell at P.

The solid angle subtended at P by the shell ABC (going off to infinity in the direction AC) is the solid angle between the planes of which AP and PE are the traces. Suppose a sphere with centre P and unit radius be constructed, the solid angle ω is given by—

$$\omega = 4\pi \frac{\text{Area of sphere intercepted by the planes AP and PE}}{\text{Total area of the sphere}}$$

$$= 4\pi \frac{2\left(\frac{\pi}{2} - a\right)}{4\pi};$$

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that is—
$$\omega = \pi - 2a;$$

where
$$\tan a = \frac{x}{y}.$$

Hence
$$\omega = \pi - 2\tan^{-1}\frac{x}{y}.$$

The potential at P due to the shell is therefore—

$$V = \Phi\omega = \Phi\left(\pi - 2\tan^{-1}\frac{x}{y}\right).$$

The force at P in any direction is given by the rate of diminution of the potential V in that direction (see § 98). The force in the direction AD is therefore—

$$F_x = -\frac{dV}{dx},$$

and the force in the direction DP is—

$$-\frac{dV}{dy},$$

and hence the force *in the direction PD* is—

$$F \quad -\frac{dV}{dy}.$$

Hence
$$F_x = -\frac{d}{dx}\left(\Phi\omega\right) = +\frac{d}{dx}\left(2\Phi\tan^{-1}\frac{x}{y}\right)$$
$$F_y = +\frac{d}{dy}\left(\Phi\omega\right) - +\frac{d}{dy}\left(2\Phi\tan^{-1}\frac{x}{y}\right).$$

That is—
$$F_x = 2\Phi\frac{y}{x^2+y^2}: \qquad F_y = 2\Phi\frac{x}{x^2+y^2};$$

and the total force at P is—

$$F = \sqrt{F_x^2 + F_y^2} = 2\Phi\left(\frac{1}{\sqrt{x^2+y^2}}\right).$$

That is—
$$F = \frac{2\Phi}{r},$$

where r is the distance of P from A.

It will be seen in § 185 that if C is the strength of the current in the conductor A, measured in electro-magnetic units, $C = \Phi$, and the force on unit magnetic pole at P due to the current in A will be—

$$F = \frac{2C}{r} \text{ dynes},$$

and this is true whether the medium to which the conductor is placed is magnetic or non-magnetic so long as the medium extends indefinitely in all directions (see also § 112).

Now in Fig. 212 $$\tan \beta = \frac{F_y}{F_x} = \frac{x}{y},$$

and hence $\beta = \alpha$, and the force F is therefore at right angles to AP. Since the force has the same magnitude at all points, at a given distance r from A, it follows that *the lines of force are circles concentric with the conductor* A.

In Fig. 213 some of the lines of force are shown for the case of a very long conductor at right angles to the plane of the paper, and in which the current is flowing *away* from the observer as in Figs. 211 and 212.

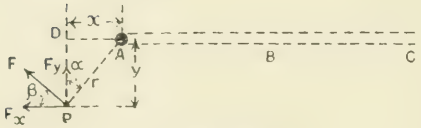


FIG. 212.

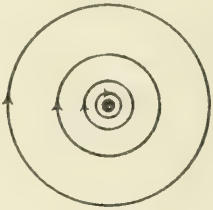


FIG. 213.

It will be seen that the force at the point P due to the current in the conductor A (Figs. 211 and 212) is the same whatever the position of the shell ABC may be, provided the boundary of the shell coincides with the conductor A and extends to infinity in other directions.

183. Biot's and Savart's Experiment.—The magnetic force due to a current in a long straight conductor may be examined by means of a small compass needle, as was done first by Biot and Savart.

In Fig. 214 let A represent the conductor arranged vertical to the plane of the paper and in which a current flows *towards* the observer. The lines of force due to this conductor will be concentric circles, and the direction of the force will be counter-clockwise direction as given by the rule stated in § 180. Suppose NS represents the magnetic meridian, and *ns* a small compass needle set with its axis on the east-west line through the conductor A.

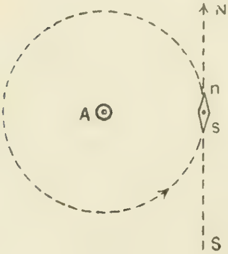


FIG. 214.

If the compass needle be allowed to oscillate under the action of the earth's field alone—that is, for the condition in which no current flows in the conductor A—the time of a small oscillation will be (§ 126)—

$$T_\circ = 2\pi\sqrt{\frac{K}{MH}},$$

where K is the moment of inertia of the needle,
,, M ,, magnetic moment ,, ,,
and H is the horizontal component of the intensity of the earth's field.

The number of oscillations per second will therefore be—

$$f_\circ = \frac{1}{T_\circ} = \frac{1}{2\pi}\sqrt{\frac{MH}{K}}.$$

Suppose the needle *ns* be first placed at a distance r_1 cms. from the conductor A, and suppose a current flows in the conductor in the direction towards the observer, the number of oscillations per second of the needle will be—

$$f_1 = \frac{1}{2\pi}\sqrt{\frac{M(H+F_1)}{K}},$$

where F_1 is the intensity of the magnetic force due to the current at the distance r_1.

Let the needle now be moved, with its axis on the east-west line through the conductor A, to a distance r_2 from the conductor, the current in the conductor being kept of constant magnitude. The number of oscillations per second of the needle will now be—

$$f_2 = \frac{1}{2\pi}\sqrt{\frac{M(H+F_2)}{K}},$$

where F_2 is the intensity of the magnetic force due to the current at distance r_2.

Experiment shows that—

$$\frac{F_1}{F_2} = \frac{f_1^2 - f_\circ^2}{f_2^2 - f_\circ^2} = \frac{r_2}{r_1};$$

and hence the force due to the current varies inversely as the distance from the conductor.

184. Maxwell's Experiment.—Another experimental proof that the force due to the current in a long straight conductor is inversely proportional to the distance from the conductor was devised by Maxwell.

The conductor is arranged in a vertical position as indicated by A in Fig. 215. A light carriage of non-magnetic material is suspended so as to be free to turn round the conductor as an axis. It is found that when a magnet is secured to this carriage and a current flows in the wire A there is no couple tending to turn the carriage round the conductor, and this is true whatever the position of the magnet on the carriage may be.

Assume, for example, that a thin uniform bar magnet NS in Fig. 215 is used. The magnet may be regarded as composed of two equal and opposite magnetic poles fixed at its extremities. Let F_n and F_s be the forces on the poles, each force acting at right angles to the plane through the conductor and the respective pole. If r_1 and r_2 are the distances of the poles from the conductor, the couple tending to turn the magnet (and hence also the carriage) round the conductor will be—

$$F_n r_1 - F_s r_2.$$

But experiment shows that this resultant couple is zero.

Hence $$\frac{F_n}{F_s}=\frac{r_2}{r_1};$$

or, the forces are inversely proportional to the distances of the poles from the conductor.

185. Magnetic Force at a Point on the Axis due to a Current in a Plane Circular Coil.—In Fig. 216 is represented a single turn circular coil arranged perpendicular to the plane of the paper. A current flows in a counter-clockwise direction, as viewed by an observer at P, as is denoted by the respective signs ⊙ and ⊕ in Fig. 216. The magnetic potential at any point P on the axis of the coil and due to the current is given by (§ 103)

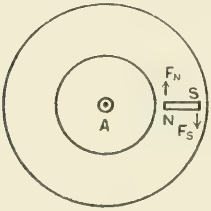


Fig. 215.

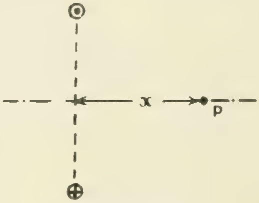


Fig. 216.

$$V=\Phi\omega,$$

where ω is the solid angle subtended at P by the coil and Φ the strength of the magnetic shell which is equivalent to the current in the coil, and of which the boundary coincides with the coil.

For the given direction of the current in the coil the positive side of the equivalent shell will face P.

A little consideration will show that the solid angle ω subtended at P by the coil is—

$$2\pi\left(1-\frac{x}{\sqrt{R^2+x^2}}\right),$$

where x cms. is the distance of P from the centre of the coil, and R cms. is the radius of the coil.

Hence the potential at P is—

$$V=\Phi 2\pi\left[1-\frac{x}{\sqrt{R^2+x^2}}\right].$$

The force at P is—

$$F=-\frac{dV}{dx}=-2\pi\Phi\left[\frac{x^2}{(R^2+x^2)^{\frac{3}{2}}}-\frac{1}{\sqrt{R^2+x^2}}\right];$$

that is—

$$F=2\pi\Phi\frac{R^2}{(R^2+x^2)^{\frac{3}{2}}}\text{ dynes.}$$

If P is at the centre of the coil, $x=o$, and the force on unit pole at P is—

$$F=2\pi\frac{\Phi}{R}\text{ dynes.}$$

If the radius of the coil is 1 cm. and the strength Φ of the shell is unity, the force at the centre of the coil is 2π dynes.

This result leads to the definition of the *electro-magnetic unit of current strength*—viz.:

> **The electro-magnetic unit of current strength is such that when it flows in a plane single-turn circular coil of 1 cm. radius it acts with a force of 2π dynes on a unit magnetic pole placed at the centre of the coil.**

The method of measuring a current in terms of this electro-magnetic (*i.e.*, absolute) unit is explained in § 297.

The electro-magnetic unit of current is too large for practical purposes, and hence the practical unit of the **ampère** is employed, the relation of the two units being—

10 ampères = 1 electro-magnetic unit.

It should therefore be remembered that the ampère has been arbitrarily chosen as $\frac{1}{10}$ of the electro-magnetic unit. The coulomb (§§ 8, 132) is thus also an arbitrary unit, a coulomb being defined as such that if a current of 1 ampère flows across the section of a conductor a quantity of electricity of 1 coulomb crosses the section per second.

From what has been said above, it will be seen that if a current of strength 1 electro-magnetic unit flows in a coil, the force due to this current is the same as the force due to a magnetic shell of unit strength, and of which the boundary coincides with the conductor. Hence, if a current of C electro-magnetic units flows in the coil the equivalent shell has a strength $\Phi=C$.

Hence the force at a point P due to a current of strength C electro-magnetic units (or i ampères) flowing in the long straight conductor (Fig. 214) is by § 182—

$$F=\frac{2C}{r}\text{ dynes}=\frac{2i}{r10}\text{ dynes,}$$

where r cms. is the distance of the point P from the conductor.

If a current of strength C flows in a single-turn circular coil of 1 cm. radius, the force at the centre of the coil will be $2\pi C$ dynes—that is, the force which would be experienced by a unit pole at the centre of the coil would be $2\pi C$ dynes. Now the field due to unit pole at the centre of the coil has unit intensity at the place occupied by the conductor, and is directed radially—*i.e.*, perpendicular to the conductor. Hence by § 105 it follows that the force on an element δs of the conductor may be written—

$$\delta s C.$$

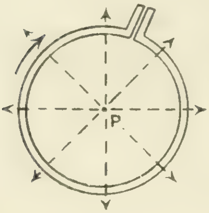


Fig. 217.

If the current flows in a clockwise direction round the coil, as shown in Fig. 217, the force will be such, if the unit pole is a positive pole, as to tend to move the coil parallel to itself and towards the observer. The total force on the coil is—

$$\int_0^{2\pi} C\,ds = 2\pi C,$$

which is, of course, the same as the force exerted by the coil on the unit pole.

186. Laplace's Formula.—In § 105 it was shown that the force experienced by a magnetic shell when placed in a magnetic field would be accounted for if a certain force were applied to each element of the boundary of the shell. From Ampère's theorem of the equivalence of a current in a coil and a magnetic shell, the same formula can be used to express the force on each element of a conductor in which a current flows and which is placed in a magnetic field.

This is Laplace's formula, and may be restated as follows:

The force on each element δs of a conductor in which a current of C electro-magnetic units flows and which is placed in a magnetic field of intensity H is given by—

$$F = CH\delta s \sin a \text{ dynes},$$

where a is the angle which the element δs makes with the direction of the field H. The direction of this force is perpendicular to the field and to the element δs.

Thus, if in Fig. 218 the direction of the current in δs and the direction of the field H are as marked, the direction of the force F will be towards the observer.

The direction of the force, when the directions of the current and field are given, may be found by the following rule, due to Fleming, and known as the Left-hand Rule (see also p. 280):

Place the thumb, forefinger, and middle finger of the left hand mutually at right angles. If the forefinger points in the direction of the magnetic field in which the conductor is placed, and if the middle finger points in the direction of the current, then the thumb will point in the direction of the force acting on the conductor.

The direction of the force which acts on a current-carrying conductor when placed in a magnetic field may be considered in a slightly different manner as follows:

In Fig. 219 the lines of force of a magnetic field are represented, and in Fig. 220 the lines of force due to a current in a conductor perpendicular to the plane of the paper are represented, the direction of the current being away from the observer as indicated by ⊕. If the conductor of Fig. 220 be placed in the field of Fig. 219, the two fields being superposed gives a field distribution round the conductor somewhat as shown in Fig. 221.

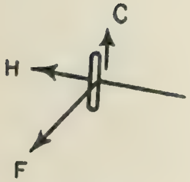


Fig. 218.

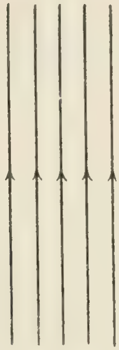


Fig. 219.

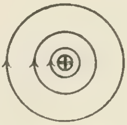


Fig. 220.

The magnetic lines of force are in equilibrium under the action of a force along the lines and a pressure normal to the lines, as was shown to hold for the corresponding electrostatic case (§ 61).

The lines of force in Fig. 221 being crowded to the left of the conductor tend to straighten out and thereby exert a force F on the conductor directed to the right.

187. Field due to a Current in a Plane Circular Coil.—In § 185 the intensity of the magnetic force at a point P on the axis of the coil was found from a consideration of the solid angle subtended by the coil at P. The intensity of the force at any point not on the axis is similarly determined from a knowledge of the solid angle subtended by the coil at the point. The expression, however, for the solid angle subtended by the coil at a point which is not on the axis is a somewhat complicated one and will not be considered here. In Fig. 222 are shown the lines of force due to such a coil in a diametrical plane perpendicular to the plane of the coil.

188. Work done in moving a Current-carrying Conductor in a Magnetic Field.—First suppose the magnetic field to be of uniform intensity H, and the conductor placed in a plane perpendicular to the field, as shown in Fig. 223.

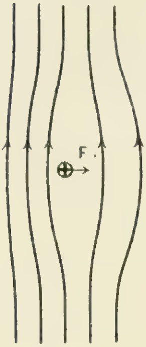


FIG. 221.

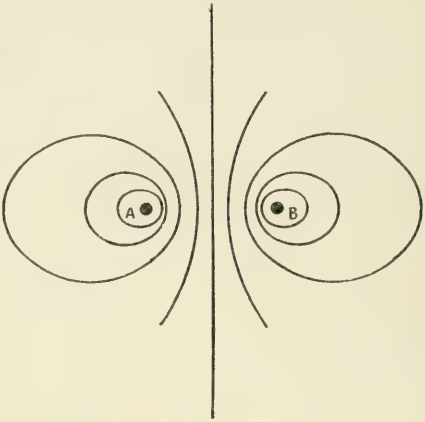


FIG. 222.

The force on the conductor will be (§ 186)—

$$HlC \text{ dynes—that is, } Hli10^{-1} \text{ dynes,}$$

where l cms. is the length of the conductor and the current strength is C in electro-magnetic units and i in ampères.

Suppose the conductor be allowed to move under the action of this force through a distance d cms. The work done on the conductor will be—

$$Hl\frac{i}{10}d \text{ ergs.}$$

The quantity Hld represents the total number of unit magnetic lines cut by the conductor during the movement. It is to be noted that, although the magnetic field of intensity H is superposed on the magnetic field due to the current, thus producing a distorted resultant field, as explained with reference to Figs. 219, 220, 221, the lines due to the current are not cut by the conductor during the movement, since they move with the conductor. Hence the work done during the movement is given by the expression:

$$\frac{\text{Total number of lines cut} \times \text{current in ampères}}{10} \text{ ergs.}$$

(A more general statement of the force on a conductor which carries a current and is placed in a magnetic field would be obtained by putting B instead of H in the above formulæ (see also p. 251). For magnetic fields in air, however, $B = H$.)

As will be seen later, the energy necessary to perform this work is derived from the battery or other source supplying the current, since the movement of the conductor in the field induces an e. m. f. in the conductor, which opposes the flow of current, and hence, in order to keep the current strength constant during the movement, the p. d. across the conductor must be increased correspondingly (see § 200, Chapter XV.).

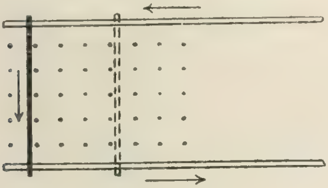


FIG. 223.

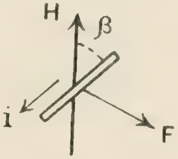


FIG. 224.

If the conductor is not perpendicular to the direction of the field H, but is inclined at an angle β (see Fig. 224), the force on the conductor is $\frac{Hli}{10}\sin\beta$ dynes, and the work done by a movement of the conductor of d cms. in the direction of this force is $\frac{Hlid\sin\beta}{10}$ ergs. But $Hld\sin\beta$ is the number of lines cut by the conductor during the movement, and hence the work done is again given by the number of lines cut multiplied by $\frac{1}{10}$ of the current in ampères.

If the field is not of uniform intensity the same result is arrived at by considering elementary portions of the conductor moving through small fractions of the distance, so that the field swept out by each element of the conductor during each stage of the movement may be taken as constant.

Hence, *if a conductor carrying a current i ampères cuts N unit magnetic lines of induction, the work done during the movement is*—

$$N\frac{i}{10} \text{ ergs.}$$

189. Mutual Force between Two Long Parallel Conductors in which a Current flows.—Suppose in Fig. 225 the two conductors are A and B, each carrying a current of i ampères, and distant r cms. apart. The intensity of the field at B due to the current in A is (§§ 185 and 189)—

$$\frac{2i}{10r} \text{ lines per sq. cm.}$$

The force on the conductor B carrying a current of i ampères and situated in a field of intensity $\frac{2i}{10r}$ is (§ 188)—

$$\frac{2i^2}{10r} \text{ dynes per cm. length of conductor.}$$

It is clear that there must also be an equal force on conductor A.

Hence the total force on each conductor is—

$$\frac{2i^2}{10r}\, l \text{ dynes,}$$

where l cms. is the length of the conductor.

If the current in each conductor is in the same direction, the force between the conductors is one of attraction. If the current is in opposite directions in the two conductors (*e.g.*, as in dynamo mains), the force between the conductors is one of repulsion.

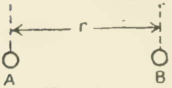


Fig. 225.

Example.—If a current of 100 ampères flows in each conductor, and the conductors are 3 inches apart and 100 yards long, the force between them will be—

$$\frac{2\times10^4\times3600\times2{\cdot}54}{10\times7{\cdot}62} \text{ dynes}$$

$$=2{\cdot}4\times10^6 \text{ dynes}$$

$$=5{\cdot}4 \text{ lbs. weight.}$$

190. Field due to Two Parallel Coils in each of which the Same Current flows.—From § 185 it is seen that if a current of strength 1 electro-magnetic unit (*i.e.*, 10 ampères) flows in a single turn coil of radius R cms., the force acting on unit magnetic pole at a point on the axis of the coil and distant x cms. from the centre of the coil is—

$$F=\frac{2\pi R^2}{(R^2+x^2)^{\frac{3}{2}}} \text{ dynes.}$$

If the values of F be found for various values of x, it will be seen that the force is a maximum at the centre of the coil, and as x increases the force falls off at a rate which is a maximum when $x=\frac{R}{2}$, as will be proved by putting $\frac{d^2F}{dx^2}=0$.

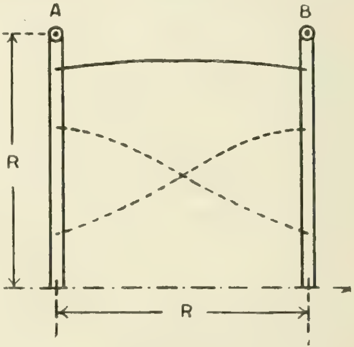


Fig. 226.

Now for certain purposes (see p. 304) it is necessary to produce a magnetic field of as uniform an intensity as possible over an appreciable space, and this may be achieved by means of two coils as follows:

Suppose two coils A and B (Fig. 226), each of radius R cms., be arranged parallel to each other, and so that the distance between the mean planes of the coils is equal to the radius of the coil (*i.e.*, R cms.), and if the same current flows round each coil in the same direction the force due to each coil at points along the common axis is shown by the ordinates of the two

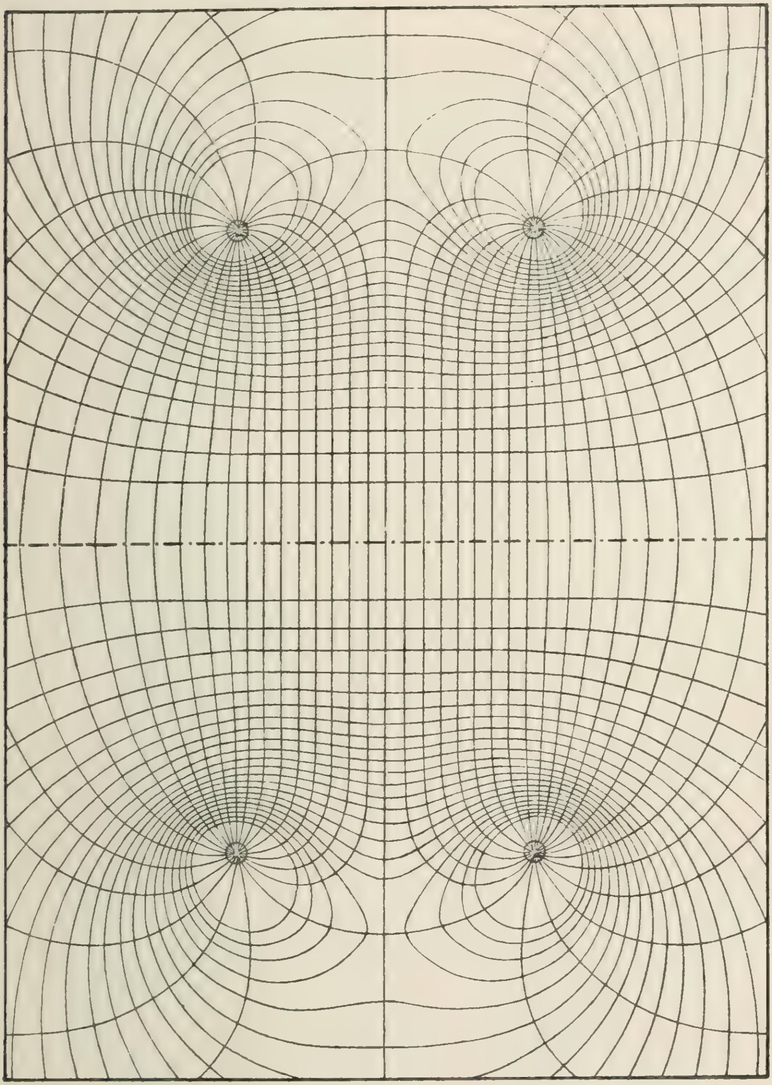


FIG. 227.
(By permission of the Delegates of Clarendon Press, Oxford.)

dotted curves. The resultant force due to the two coils is given by the ordinates of the full line curve. It will be seen that the resultant force is almost absolutely constant over a considerable range on each side of the point, midway between the coils. If the radius of each coil is 20 cms. and the distance between the coils is also 20 cms., the resultant force at a point on the common axis midway between the coils is—

$$\frac{2\times 2\pi\times 20^2}{(20^2+10^2)^{\frac{3}{2}}}=0{\cdot}45 \text{ dynes,}$$

assuming, as stated above, that each coil has only one turn. If each coil has $w/2$ turns and a current of i ampères flows round each coil, the resultant force at a point on the common axis midway between the coils is—

$$\frac{0{\cdot}45\, wi}{\text{R}} \text{ dynes.}$$

A practical application of this arrangement is found in Helmholtz' tangent galvanometer (see also pp. 304 and 305).

In Fig. 227 are given the lines of force embracing two such parallel coils placed at a distance apart, equal to the radius of each coil, the lines of force being drawn in a plane containing the common axis of the two coils.

The lines terminating on the conductors are equipotential lines, and therefore cut the lines of force at right angles.

It is clear from this diagram that the intensity of the force is practically constant over a comparatively large region round the central point, and midway between the coils.

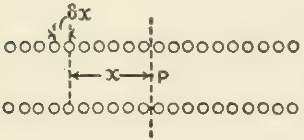


Fig. 228.

191. The Magnetic Force at the Middle of a Long Solenoid.—Let the solenoid be wound with w turns per cm. length, and let l cms. be the length of the solenoid.

The magnetic force at the point P (Fig. 228) due to the current in the element δx of the solenoid distant x from the middle point P is (by § 185)—

$$2\pi\text{C}\,\frac{\text{R}^2}{(\text{R}^2+x^2)^{\frac{3}{2}}}\,w\delta x,$$

where C is the current strength in electro-magnetic units and R is the radius of the turns of the solenoid.

The total force at P due to the whole solenoid is therefore—

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} 2\pi\text{C}\,\frac{\text{R}^2}{(\text{R}^2+x^2)^{\frac{3}{2}}}\,w\,dx = 2\pi\text{C}w\left[\frac{x}{(\text{R}^2+x^2)^{\frac{1}{2}}}\right]_{x=-\frac{l}{2}}^{x=\frac{l}{2}}$$

$$=2\pi\text{C}w\,\frac{l}{\left(\text{R}^2+\frac{l^2}{4}\right)^{\frac{1}{2}}} \text{ dynes.}$$

If R is small compared with l the force acting on unit magnetic pole at P will be—

$$4\pi\text{C}w \text{ dynes.}$$

That is, the intensity of the magnetic field at P is—

$$H = 4\pi C w \text{ lines per sq. cm. (or gauss).}$$

This result is of considerable importance in connection with the calibration of galvanometers.

192. Magneto-motive Force round a Closed Curve linking a Current Carrying Circuit.—It has been shown in § 188 that when a conductor in which a current of i ampères is flowing is cut by N unit magnetic lines of force the work done is—

$$N \frac{i}{10} \text{ ergs.}$$

It is a matter of indifference whether the conductor cut the lines by reason of the flux moving, or the conductor moving, or both conductor and flux moving.

Now consider the electric circuit shown in Fig. 229 in which a current of i ampères is flowing, and assume a unit magnetic pole be carried once round the closed circuit linked with the electric circuit and shown dotted in Fig. 229. From § 89 it is known that the unit pole gives rise to 4π unit lines, and if the pole is carried round the closed loop linked with the electric circuit each of the 4π lines will cut the electric circuit once. Hence the work done in carrying the unit pole once round the closed loop is—

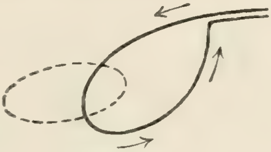


FIG. 229.

$$4\pi \frac{i}{10} \text{ ergs.}$$

If the electric circuit has w turns in series and the closed loop links the whole of the w turns, the work done is—

$$\frac{4\pi}{10} iw \text{ ergs,}$$

because it is clear that i ampères in w turns will be equivalent to wi ampères in one turn.

The work done is independent of the actual path taken in making the journey round the closed loop so long as the path only links the circuit once.

The work done in traversing the closed loop with unit pole is the *magneto-motive force* round the loop, and is usually written m. m. f.

Hence the magneto-motive force round a closed loop is $\frac{4\pi}{10}$ (ampère-turns) linking the loop.

This is an extremely important result in connection with the design of the magnetic circuit of electrical machines and apparatus.

Now the work done in carrying unit pole round the magnetic circuit may also be stated as the sum—

$$\Sigma Hl,$$

where H is the intensity at any part of the circuit and l cms. the length of the part of the circuit over which the force has the value H.

Hence $$\frac{4\pi}{10}wi = 1\cdot257wi = \Sigma Hl;$$

or $$wi = 0\cdot8\Sigma Hl.$$

In order to find the value of the flux produced by any given value of the ampère turns wi, it is necessary to plot a curve connecting flux and ampère-turns by assuming different values for the flux and calculating the corresponding values of $0\cdot8\Sigma Hl$.

If the path of the closed loop is not wholly in air but passes partly or wholly through magnetic materials, the results stated above still hold because the work done in carrying a unit pole round a closed path in a field due to *magnetised* bodies is zero, just as the work done in carrying a unit charge round a closed path in an electrostatic field is zero (see also § 98).

Hence, whatever materials the closed loop may pass through, the magneto-motive force is $\frac{4\pi}{10}$ × ampère-turns linked with the loop.

Example.—As an example of the application of this result the magnetic circuit of a two-pole dynamo will be considered, as shown in Fig. 230. The poles are marked N and S, and on each pole is wound an exciting coil. The magnetic flux passes from the N pole, across the air-gap, through the armature, across the second air-gap, through the S pole, and divides into the two halves of the yoke, thus completing the magnetic circuit to the N pole.

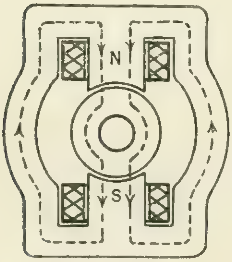


FIG. 230.

The total flux of magnetic induction is constant at any cross-section of the circuit (neglecting any leakage which passes from the sides of the poles into the sides of the yoke).

The induction density at any cross-section of the circuit is obtained by dividing the magnitude of the flux by the cross-section. Having thus determined the value of the induction B at different parts of the circuit, the corresponding values of H are read off from the B-H curves given in Chapter VIII., Fig. 131.

Then, if i is the current in ampères in the exciting coils and w the total number of turns in series, the m. m. f. round the magnetic circuit is—

$$\frac{4\pi}{10}wi = 1{\cdot}257wi.$$

For this example it is assumed that the cross-section of the magnetic circuit in poles and yoke is the same.

The cross-section of the yoke is the sum of the sections of the two limbs through which the total pole flux passes.

The following numerical values have been assumed, viz.:

Area of air-gap, $A_g = 100$ sq. cms.

Length of double air-gap, $l_g = 0{\cdot}4$ cms.

Total cross-sectional area of (iron) armature, $A_a = 75$ sq. cms.

Length of the mean path of the flux in the armature, $l_a = 17$ cms.

Cross-sectional area of (steel) poles and yoke, $A_y = 50$ sq. cms.

Total length of the mean path of the flux in poles and yoke, $l_y = 88$ cms.

The area of the gap is taken as being twice the area of the poles, it being assumed that pole shoes are fitted but are not shown in Fig. 230.

In the following table, a series of values of total flux N is taken and the corresponding values of the ampère-turns calculated.

Total Flux N	$B_g = H_g = \frac{N}{A_g}$	$B_a = \frac{N}{A}$	H_a	$B_y = \frac{N}{A_y}$	H_y	$H_g l_g$	$H_a l_a$	$H_y l_y$	ΣHl	$wi = 0{\cdot}8\Sigma Hl$
100,000	1,000	1,330	0·6	2,000	2·2	400	10	200	610	488
200,000	2,000	2,660	1	4,000	2·9	800	17	255	1,072	858
300,000	3,000	3,990	1·2	6,000	3·6	1,200	20	320	1,540	1,232
400,000	4,000	5,320	1·5	8,000	4·9	1,600	25	430	2,055	1,644
500,000	5,000	6,650	1·7	10,000	6·6	2,000	29	580	2,610	2,088
600,000	6,000	7,980	2	12,000	9·5	2,400	35	840	3,275	2,620
700,000	7,000	9,310	2·5	14,000	15	2,800	42	1,320	4,162	3,332
800,000	8,000	10,640	3	16,000	27·2	3,200	51	2,400	5,651	4,521

The results are plotted as a curve in Fig. 231, from which it is possible to read off the value of the total flux for any given value of the exciting ampère-turns.

193. Magnetic Reluctance.—If N is the flux of induction in a magnetic circuit which consists of different media it has been seen that the magneto-motive force is given by the expression—

$$\text{m. m. f.} = \frac{4\pi}{10}wi = (H_1 l_1 + H_2 l_2 + H_3 l_3 \ . \ . \ .),$$

where H_1 is the intensity of the magnetic force along a path of length l_1 cms. and similarly for H_2; H_3. . . .

But $$H_1=\frac{B_1}{\mu_1}=\frac{N}{A_1\mu_1},$$

where A_1 sq. cms. is the cross-sectional area of path of l_1.

Similarly $$H_2=\frac{N}{A_2\mu_2}. \; . \; . \; .$$

Hence $$\text{m. m. f.}=\frac{4\pi}{10}wi=N\left[\frac{l_1}{A_1\mu_1}+\frac{l_2}{A_2\mu_2}+\frac{l_3}{A_3\mu_3}+\; . \; . \; .\right].$$

From a comparison of this equation with the analogous one for the electric circuit, viz.:

e. m. f. = current × resistance,

the quantity $$\left[\frac{l_1}{A_1\mu_1}+\frac{l_2}{A_2\mu_2}+\frac{l_3}{A_3\mu_3}+\; . \; . \; .\right]$$

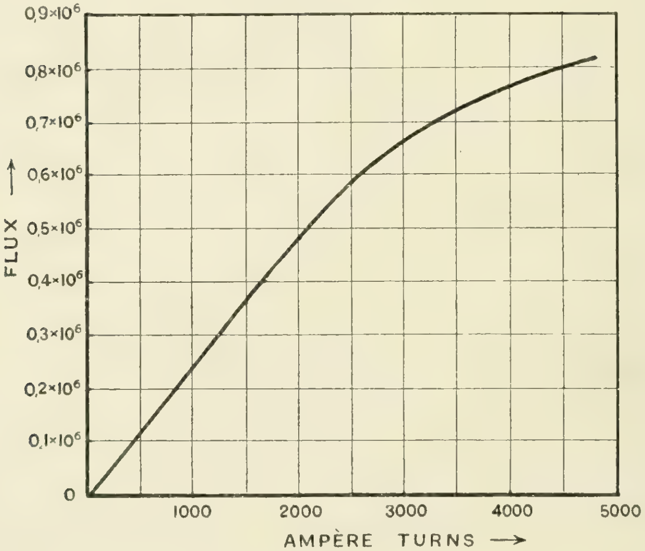


FIG. 231.

has been termed the *reluctance* of the magnetic circuit, and consequently the following relationship holds, viz.:

m. m. f. = flux × reluctance.

The reluctance of *any part* of a magnetic circuit of which the length is l cms., the cross-section A sq. cms., and the permeability is μ, is—

$$\frac{l}{A\mu}.$$

For example, consider an electro-magnet, as shown in Fig. 232.

If l_1 cms. is the length of the magnetic circuit in the iron and l_2 cms. the length of the gap, and if A sq. cms. is the area of the circuit—

$$\frac{4\pi}{10} wi = \mathrm{N}\left[\frac{l_1}{\mathrm{A}\mu_1} + \frac{l_2}{\mathrm{A}}\right],$$

the permeability of the air-gap being unity.

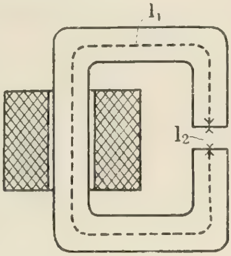


FIG. 232.

For values of flux density in the iron below, say B=15000 (see Fig. 135, p. 418) the value of μ_1 will be large and $\frac{l_1}{\mathrm{A}\mu_1}$ will be relatively negligibly small as compared with $\frac{l_2}{\mathrm{A}}$, and hence the m. m. f. is almost entirely used in driving the flux across the air-gap.

CHAPTER XV

ELECTRO-MAGNETIC INDUCTION

194. Faraday's Experiments.—Faraday, in looking for an effect in electro-magnetic magnetism analogous to electro-static induction (§ 3), made in the year 1831 an experiment as follows:

In Fig. 233 AB is a stretched wire connected to a galvanometer G, so that a closed circuit is obtained. A wire CD is arranged parallel to AB and connected through a switch to a Daniell's cell. When a steady current flows in the wire CD no deflection of the galvanometer needle is obtained. Faraday discovered, however, that when the switch is closed the needle of the galvanometer G shows a momentary deflection and then comes to rest. When the switch is opened the needle of G again shows a momentary deflection and then comes to rest, the deflection when opening the switch being in the opposite direction to that obtained when closing the switch. A current is therefore induced in the circuit of AB only when the electrical condition of the neighbouring circuit CD is disturbed—*i.e.*, when the current in CD is changed.

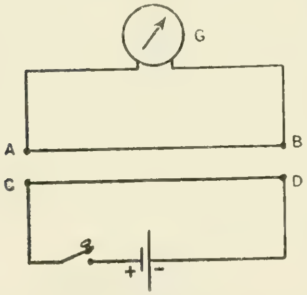


Fig. 233.

If the current in CD flows from C to D when the switch is closed, the deflection of the needle of G corresponds to a current in AB flowing from B to A. When the switch K is opened (*i.e.*, the current in CD stopped), the deflection shown by the galvanometer G corresponds to a current in AB flowing from A to B.

This electro-magnetic induction effect between the two circuits is due to the variation in the magnetic flux in the neighbourhood of the wires when the current is altered. It was shown in § 180 that when a current flows in a wire such as CD in Fig. 233, a magnetic field is established in the neighbouring space, the lines of magnetic force being circles concentric with the wire. When the current is started in CD the magnetic field may be considered as becoming established by the lines of force expanding outwards from the wire CD, linking the circuit of which AB is a part, *and thus generating an e. m. f. in the circuit of AB.* When the current in CD has become steady the magnetic field associated with this current becomes steady, and the condition is then reached that a magnetic flux is linked with the circuit of AB. So long as this flux is steady no e. m. f. is induced in the circuit of AB.

When the current in CD is stopped, the magnetic field due to the current disappears, and this may be considered as taking place by the lines of force shrinking and eventually collapsing on to the conductor CD.

In thus shrinking the flux linked with the circuit of AB is withdrawn, and *during the withdrawal of the flux an e. m. f. is induced in the circuit of AB*, the direction of this e. m. f. being opposite to that induced when the magnetic field was being established—viz., when the lines of force were expanding.

Further, by revolving a given circuit at different speeds in a given magnetic field Faraday arrived at the following result:

> **In a circuit any part of which moves across lines of magnetic induction, an e. m. f. is induced which is proportional to the rate at which the number of lines linked with the circuit changes.**

195. Lenz's Law.—Almost immediately after Faraday's discovery Lenz gave the rule by which the *direction* of the induced e. m. f. may be determined, and this rule is known as Lenz's Law—viz.:

> **The direction of the induced e. m. f. is always such that by its electro-magnetic action it tends to oppose the effect which produces it.**

Thus in Fig. 233, when the switch is closed and the current started in CD, an e. m. f. is induced in AB in a direction such that it tends to prevent the magnetic field due to the current in CD from linking the circuit of AB, hence the current induced in AB will be in the opposite direction to that started in CD. When the switch is opened the stoppage of the current in CD will induce an e. m. f. in the circuit of AB in a direction such that it will tend to oppose the withdrawal of the flux linked with AB, hence the current induced in AB will be in the same direction as the current which flowed in CD before opening the switch K.

196. Further Examples of Electro-Magnetic Induction.—In Figs. 234 and 235 NS is a permanent magnet and C is a closed coil of wire. If the coil C is threaded quickly over the N pole of the magnet (Fig. 234), a current will be induced in the coil during the movement, the direction being such that it will oppose the flux of the magnet from becoming linked with the coil.

If the coil is now quickly withdrawn from the position shown in Fig. 235 a current will be induced in the reverse direction to that shown in Fig. 234, and Lenz's Law shows that the direction of the current induced in the coil during the withdrawal will be as shown in Fig. 235, whether the coil be removed by drawing it off the N end or the S end of the magnet.

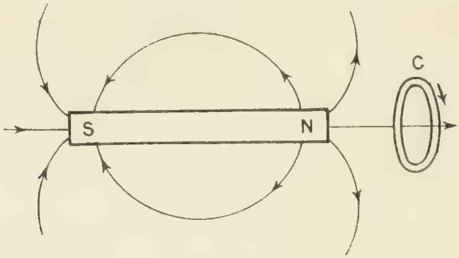


Fig. 234.

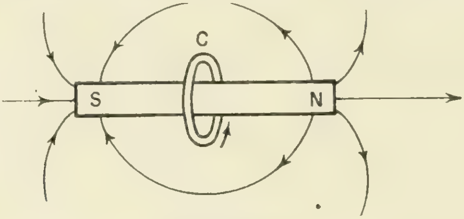


Fig. 235.

In Fig. 236 a coil A is connected to a galvanometer, and a coil B wound on an iron core and connected to a battery. If the coil and iron core B is inserted in the coil A, a current will be induced in A during the movement, and if the coil and iron core B is then withdrawn a current will be induced in A in the opposite direction to that induced when inserting B in A. The strength of the induced currents is very greatly increased by winding the coil B on an iron core, as shown in Fig. 236, since the flux due to a current in the coil B is thereby greatly increased (see Chapter VIII.).

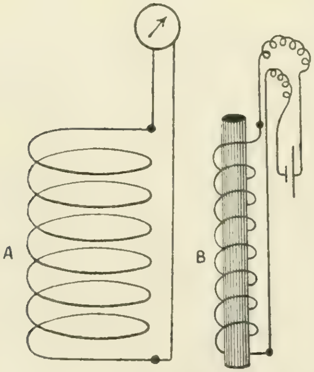


Fig. 236.

In Fig. 237 a coil is shown wound on an iron core and connected through a switch to a battery of low e. m. f. and across the terminals of the coil is connected a filament electric lamp (say, rated at 100 volts). When the switch is closed a very small current will flow in the lamp, which will therefore remain dark. When the switch is opened, however, the electro-magnetic induction is such that it tends to maintain the flux linking the coil, and an e. m. f. is thereby induced sufficiently large to cause the lamp to glow. If the lamp were removed the electro-magnetic action would be such as to cause a large spark across the switch contacts when breaking the circuit.

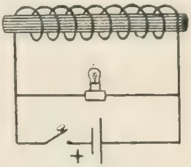


FIG. 237.

197. The Magnitude of the E. M. F. induced by Electro-Magnetic Action.—Faraday showed in 1831 the conditions under which electro-magnetic induction takes place, and Neumann in 1845 gave an expression for the *magnitude* of the induced e. m. f. The law of electro-magnetic induction may be stated as follows:

> **Whenever the flux of magnetic induction which links a circuit is changed, an e. m. f. is induced in the circuit of magnitude equal to the rate of change of the magnetic flux, and acts in such a direction as to tend to prevent that change of flux.**

It is to be noted that in so far as the induced e. m. f. is concerned it is only the relative movement of the circuit and the flux linked which matters, and it is a matter of indifference whether this relative movement corresponds to a stationary circuit or a stationary field.

In Fig. 238 let N be the flux linking the coil at any instant t. If N varies, an e. m. f. e will be induced in the coil such that—

$$e = -\frac{dN}{dt} \text{ electro-magnetic units.}$$

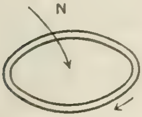


FIG. 238A.

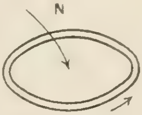


FIG. 238B.

If there are w turns in the coil closely wound so that each turn embraces the whole of the N lines, then—

$$e = -w\frac{dN}{dt} \text{ electro-magnetic units.}$$

The negative sign means that the direction of the induced e. m. f. is such that it tends to prevent the change in the number of lines linking the coil. Thus, in Fig. 238A N is assumed to be diminishing, and the induced e. m. f. produces a current tending to prevent N decreasing. In Fig. 238B N is increasing, and the induced e. m. f. produces a current tending to prevent N increasing.

The expression given above for the induced e. m. f. leads to the definition of the electro-magnetic unit of e. m. f.—viz.:

> **The electro-magnetic unit of p. d. or e. m. f. is the e. m. f. induced in a coil of one turn by a flux of magnetic induction linking the coil and changing at the rate of 1 c.g.s. unit line per second.**

This unit is too small for practical purposes, and consequently a unit one hundred million times (10^8) as large as the electro-magnetic unit is used, and is called the **volt**—that is,

$$1 \text{ volt} = 10^8 \text{ electro-magnetic units of p. d. or e. m. f.}$$

Hence an e. m. f. of one volt is induced in a coil of one turn through which the flux of magnetic induction is varying at the rate of 10^8 c.g.s. unit lines per second.

Using the same symbols as before—

$$e = -\frac{1}{10^8}\frac{dN}{dt} \text{ volts}$$

for a coil of one turn.

If the coil has w turns, each of which links the whole flux N—

$$e = -\frac{w}{10^8}\frac{dN}{dt} \text{ volts.}$$

198. Electric Generators.—In Fig. 239 let AB represent a conductor placed in the field between two poles N and S, the axis of the conductor being perpendicular to the direction of the field, as shown.

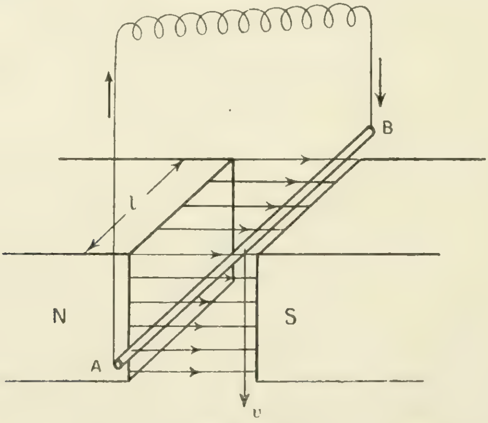


Fig. 239.

Let the conductor AB be connected to a wire of resistance R so as to form a complete circuit.

Let l cms. be the length of the pole perpendicular to the plane of the paper, and let H be the magnetic intensity of the field between the poles. Since the field is in air, the lines of force and lines of induction are identical (see p. 258). Suppose the conductor AB be moved parallel to itself with a velocity v cms. per second. At any time t let x cms. be the distance from the upper edge of the pole (Fig. 240).

The flux linking the circuit of AB at the time t is therefore—

$$N = Hlx.$$

The *magnitude* of the e. m. f. induced in circuit of AB is—

$$\frac{1}{10^8}\frac{dN}{dt}\text{ volts}$$

$$=\frac{Hl}{10^8}\frac{dx}{dt}\text{ volts}$$

$$=\frac{Hlv}{10^8}\text{ volts.}$$

The *direction* of the e. m. f. is such as to tend to oppose the movement which produces it. Hence the current induced in the circuit of AB will be (see Fig. 240) directed towards the observer, because such a current will produce a flux which, superposed on the magnetic field due to NS, will give a resultant intensity which will be greater in front of the conductor and less behind the conductor than the intensity of the original field due to NS (compare Figs. 221 and 240).

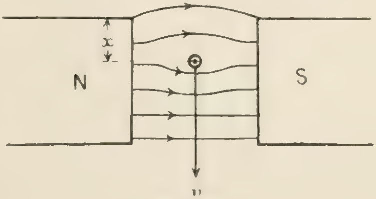


FIG. 240.

Since the magnetic lines tend to straighten out it will be clear from Fig. 240 that the magnetic field opposes the downward movement of AB. (See also p. 269, Lenz's Law.)

The same result is arrived at if it be considered that the e. m. f. is induced in the conductor when *cutting* lines of force, and may be stated as follows:

The e. m. f. induced in a conductor which cuts across lines of magnetic induction at the rate of N *per second is* $\frac{N}{10^8}$ *volts.*

This view of the induced e. m. f. being due to the lines of force *cutting* across a conductor is most convenient when considering the case of the e. m. f. induced in the conductors of electric generators and motors.

When dealing with transformers, however (§ 276), the change in flux linking the circuit is the more convenient way of dealing with the induced e. m. f.'s.

If a conductor moves in any way in a magnetic field the e. m. f. is given by the rate at which it cuts the lines of force.

Thus in Fig. 241, if the lines of force of the field are parallel to the plane of the paper and the conductor perpendicular to the plane of the paper, and if the conductor is moved parallel to itself with a velocity of v cms. per second in the direction shown in Fig. 241, the e. m. f. induced is—

$$\frac{Hlv \sin \theta}{10^8} \text{ volts,}$$

where l cms. is the length of the conductor and H the intensity of the magnetic field.

Generally stated, the magnitude of the e. m. f. induced in an element δl of a conductor moving across a magnetic field is—

$$\frac{\delta l}{10^8} \frac{dN}{dt} \text{ volts,}$$

where $\frac{dN}{dt}$ is the rate at which the element δl cuts across the lines of induction.

A very convenient rule for memorising the direction of the induced e. m. f. in a conductor which moves across a field is Fleming's "right-hand rule," thus:

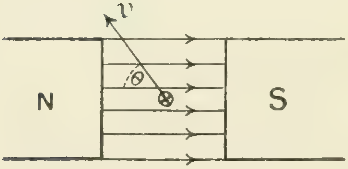


FIG. 241.

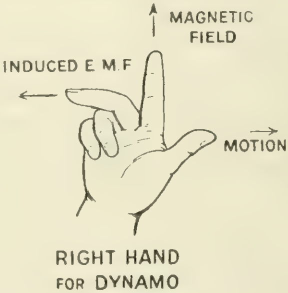


FIG. 242.

Let the thumb, forefinger, and middle finger of the right hand be placed mutually at right angles, as in Fig. 242. Let the thumb point in the direction of movement of the conductor relatively to the field, the forefinger in the direction of the field, then the middle finger will point in the direction of the induced e. m. f.

This rule is easily remembered by noting that the word thu*m*b (containing letter *m*) indicates direction of *m*ovement, *f*orefinger indicates direction of magnetic field, and the middle finger the direction of the current (usually denoted by *i*).

It is to be carefully observed that the direction of motion is the direction of movement of the conductor *relatively to the field.* Thus in Fig. 243A, if the field is stationary and the conductor moves from left to right, the induced e. m. f. is in the same direction as in Fig. 243B, in which the conductor is stationary and the field moves from right to left. In both cases the direction of movement of the *conductor relatively to the field* is from left to right.

A very useful convention for diagrammatically representing the direction of current and e. m. f. in a conductor is as follows:

If the conductor is assumed to be perpendicular to the plane of the paper, a current directed towards the observer is denoted by ⊙, which represents the point of an arrow moving toward the observer. If the current is directed away from the observer its direction is denoted by ⊕, which represents the head of an arrow moving away from the observer.

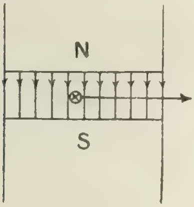


FIG. 243A.

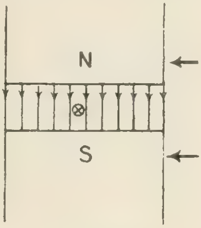


FIG. 243B.

As an example of the magnitudes of the different factors involved in generating an e. m. f. in the conductor, suppose the length of the conductor be 50 cms., the flux density H= 10000 lines per sq. cm., and the velocity of the conductor 30 metres p. s.—*i.e.*, 3000 cms. per second. Assuming the direction of motion to be at right angles both to the field and the conductor, as shown in Fig. 244, the induced e. m. f. is—

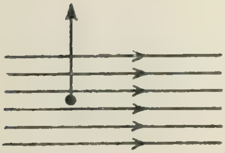


FIG. 244.

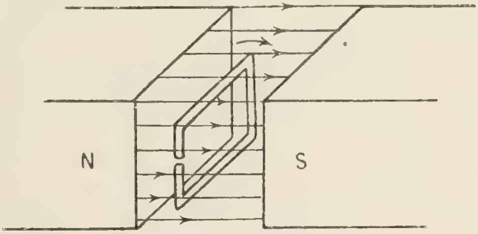


FIG. 245.

$$e=\frac{Hlv}{10^8} \text{ volts}$$

$$=\frac{10000\times 50\times 3000}{10^8} \text{ volts}$$

$$=15 \text{ volts.}$$

Suppose, now, in Fig. 245 a coil of rectangular shape is arranged so that it can rotate about an axis perpendicular to the plane of the pole sides NS, the arrangement being shown in perspective in Fig. 245, and in section in Fig. 246.

Let the coil be driven at n revs. per second, and let $2r$ cms. be the length of the side of the coil in the plane of the paper, and l cms. the length perpendicular to the plane of the paper, the area of the coil thus being $2lr$ sq. cms. If H is the intensity of the magnetic field the flux through the coil when inclined at an angle θ to the vertical is (see Fig. 246)—

$$N = 2lrH \cos \theta.$$

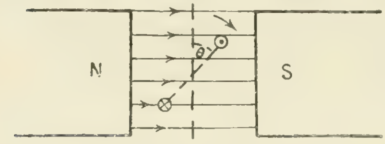


FIG. 246.

But $\theta = 2\pi nt$ radians if the time t seconds is reckoned from the moment at which the coil is vertical.

Hence $$N = 2lrH \cos 2\pi nt.$$

The *magnitude* of the induced e. m. f. is—

$$\frac{1}{10^8}\frac{dN}{dt} \text{ volts.}$$

$$= \frac{1}{10^8} 4\pi Hnlr \sin 2\pi nt \text{ volts}$$

$$= \frac{2\pi n}{10^8} N_{max} \sin \theta \text{ volts,}$$

where N_{max} is the maximum flux embraced by the coil—that is, the flux through the coil when its plane is at right angles to the field ($\theta = 0$).

It is therefore seen that the e. m. f. varies as a sine function of θ, the angular velocity of the coil being assumed constant. The e. m. f. wave plotted as a function of θ is shown in Fig. 247.

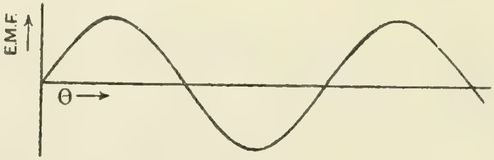


FIG. 247.

The same result will be reached if the e. m. f. be calculated from the rate at which each conductor of the loop cuts the lines of force.

In practice, in order to efficiently obtain a strong magnetic field, the coil is wound on an iron core which revolves between the pole pieces, a small gap being thus formed between the core and pole faces, as shown in Fig. 248. If the length of one of the active coil sides is l cms., the velocity is v cms. per second, and the intensity of the magnetic field is H, the e. m. f. induced in that coil side is $Hlv10^{-8}$ volts. If the core is driven at a constant speed—that is, if v is constant—it is seen that the e. m. f. is proportional to H, the flux density at that point in the gap at which the coil side is at the instant considered. The wave of e. m. f. induced in the coil in Fig. 248 will be somewhat as shown.

Instead of connecting the ends of the coil each to a separate slip ring, suppose a single ring is cut diametrically, the two parts being mounted on the axle of the core, so that they are insulated from each other. Let the two segments be connected respectively to the ends of the coil, as shown in Fig. 249.

If two brushes be arranged diametrically opposite to each other and on a horizontal or approximate horizontal axis, it will be noticed that the coil side under the N pole is connected to the brush marked –, and when the coil side passes to the S pole it is connected to the brush marked +. The negative brush is thus maintained in contact with the coil side under the N pole, and the positive brush with the coil side under the *S* pole. The polarity of each brush is thus always the same.

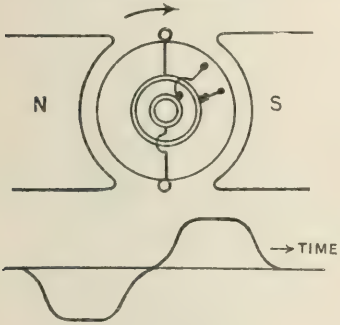


FIG. 248.

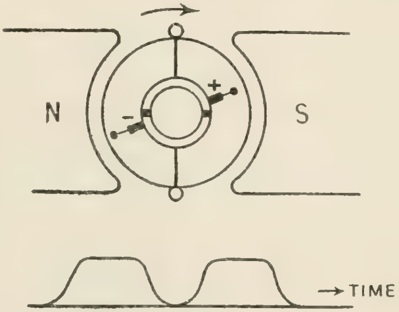


FIG. 249.

The e. m. f. developed between the brushes is as shown in Fig. 249, and is *unidirectional.*

The split ring collector device has thus rectified the e. m. f. wave shown in Fig. 248, so that the unidirectional wave of Fig. 249 is obtained, and the arrangement of Fig. 249 is thus capable of giving a current which will always be in the same direction. The split-ring collector is a very simple form of *commutator.*

A single-coil and two-segment commutator, as shown in Fig. 249, is not a practicable arrangement, since the e. m. f. varies from zero to a maximum twice per revolution of the coil.

A practical form of a direct-current generator is shown diagrammatically in Fig. 250. A number of coils wound on an iron ring and displaced relatively to each other are connected in series to form a closed circuit, the ring and coils being known as the *armature* of the machine. From each coil a tapping is brought to an insulated segment of the commutator, and the two brushes are placed diametrically opposite and on that axis which is perpendicular to the axis of the magnetic field.

For the direction of rotation of the armature and direction of field given in Fig. 250, the right-hand rule (p. 274) shows that the lower brush becomes positive and the upper brush negative. It will be seen that the coils of the armature are in two parallel branches from brush to brush.

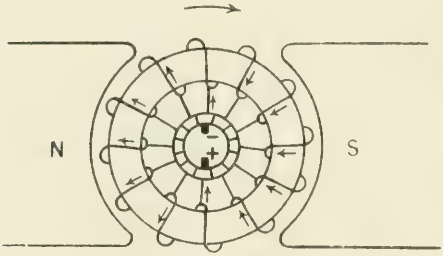


FIG. 250.

The e. m. f. induced between the brushes in such a machine may be calculated as follows:

Let N be the total flux, measured in c.g.s. unit lines, which passes from the North pole to the South pole of the field system.

Let Z be the total number of active conductors on the armature surface (*e.g.*, for the case illustrated in Fig. 250, Z=12). Further, let n be the number of revolutions made by the armature per second. It will be clear that each conductor will cut N lines in half a revolution of the armature—*i.e.*, in $\frac{1}{2n}$ second.

The mean rate of cutting of lines of force by each conductor is therefore—

$$\frac{N}{\frac{1}{2n}} = 2Nn,$$

and the mean e. m. f. induced in each conductor is (p. 273) $\frac{2Nn}{10^8}$ volts.

But there are $\frac{Z}{2}$ conductors in series between the brushes, and hence the mean e. m. f. induced between the brushes is—

$$\frac{2Nn}{10^8}\,\frac{Z}{2} = \frac{NnZ}{10^8} \text{ volts.}$$

If the armature is wound with a relatively small number of coils in series and a correspondingly small number of commutator segments, the e. m. f. induced between the brushes will pulsate considerably above and below the mean value (compare

Fig. 249). If, however, there is a relatively large number of coils in series in the armature and a correspondingly large number of commutator segments, the mean e. m. f. approaches more and more closely to the actual e. m. f. at any instant. In other words, the pulsation of the e. m. f. at the brushes becomes relatively very small.

199. Energy Involved in Generating an Electric Current.—It was shown in § 188 that in the case of a circuit in which a current of i ampères flows, if the flux linking the circuit be caused to change by an amount N c.g.s. unit lines, the mechanical work expended in producing the change of flux linking the circuit is—

$$\mathrm{N}\,\frac{i}{10}\text{ ergs}$$

$$=\frac{\mathrm{N}i}{10^8}\text{ joules,}$$

since 1 joule $=10^7$ ergs (p. 201).

If the flux linked changes at the rate of N lines per second the mechanical power expended is—

$$\frac{\mathrm{N}i}{10^8}\text{ joules per second—}i.e.\text{, watts.}$$

It has been shown, however, in § 198 that the e. m. f. induced in the circuit when the flux linking the circuit changes at the rate of N lines per second is—

$$e=\frac{\mathrm{N}}{10^8}\text{ volts.}$$

Hence the mechanical work expended in producing the change of flux through the circuit in which the current of i ampères is flowing is—

$$ei\text{ watts.}$$

If the resistance of the circuit is R ohms (see Fig. 239) the electrical power dissipated in heating the circuit is (see § 145)—

$$i^2\mathrm{R}\text{ watts.}$$

But iR is the e. m. f. in the circuit, and therefore the electrical power expended in heating the circuit is—

$$ei\text{ watts.}$$

It is thus seen that in the arrangement of Fig. 239 the mechanical power expended in driving the conductor across the field is completely accounted for by the electrical energy dissipated in the form of heat in the circuit.

200. Electric Motor.—Suppose, now, that in Fig. 251 a conductor AB is placed in a uniform field of intensity H and a current of i ampères passed through the conductor from a battery of e. m. f. E volts. If the conductor is stationary and the resistance is R ohms, then—

$$E = iR.$$

From the results of § 188 it is clear that there will be a force F acting on the conductor such that—

$$F = Hl \frac{i}{10} \text{ dynes},$$

where l cms. is the length of the conductor in the field. Suppose the conductor be free and moves with a velocity v cms. per second under the action of the force. An e. m. f. will be induced in the conductor since it is cutting across magnetic lines, and the magnitude of the e. m. f. will be by § 198—

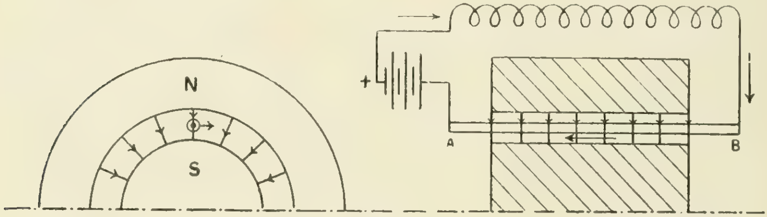


Fig. 251.

$$e = \frac{Hlv}{10^8} \text{ volts.}$$

The direction of this e. m. f. will, by Lenz's Law, be such that it will oppose the effect producing it. In this case it will reduce the current i, and hence the force F to which the movement is due. Hence the resultant e. m. f. in the circuit will be—

$$(E - e) \text{ volts},$$

and the current will be i_1 ampères, such that—

$$(E - e) = i_1 R.$$

For the relative directions of current and field, shown in Fig. 251, the direction of F will be from left to right. The direction of F may be ascertained by application of the "right-hand rule" of p. 274, it being observed that the conductor will move so that the induced e. m. f. will oppose the current.

Generally stated, if the directions of current and magnetic field are known, the corresponding direction of the force acting on the conductor may be found from Fleming's "left-hand rule," as follows:

Place the thumb, forefinger, and middle finger of the left hand mutually at right angles (Fig. 252). *If the forefinger points in the* *direction of the field and the middle finger in the direction of the current, then the thumb will point in the direction of the force which will act on the conductor—that is, the direction in which the conductor will tend to move.*

It will be seen that the "left-hand rule" applies for the conditions of a *motor*, and the "right-hand rule" for the conditions of a *generator*.

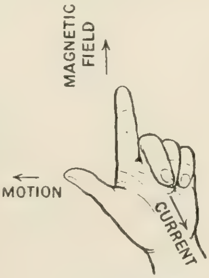
Let F_1 be the force acting on the conductor in Fig. 251 corresponding to the current i_1 ampères. The mechanical power developed will be—

$$\begin{aligned} &F_1 v \text{ ergs per second} \\ &= \frac{F_1 v}{10^7} \text{ watts (p. 201)} \\ &= \frac{H l i_1 v}{10^8} \text{ watts} \\ &= e i_1 \text{ watts,} \end{aligned}$$

and this will be the power available for performing mechanical work.

The induced e. m. f. e is termed the *back e. m. f.* of the motor. The e. m. f. corresponding to the difference between the battery e. m. f. and the back e. m. f. is necessary to drive the current i_1 through the whole circuit of battery and conductor of total resistance R.

Any direct-current generator will run as a motor if the field system is excited and current supplied to the armature. Thus in the case shown in Fig. 250, if the lower brush be connected to the positive pole of a battery, and the upper brush connected to the negative pole, current will flow in the armature in the reverse direction to that shown in Fig. 250, but the direction of rotation of the armature will be the same as that shown in Fig. 250, as will be at once seen to be the case by applying the "left-hand rule."



LEFT HAND FOR MOTOR

FIG. 252.

The mechanical power developed by the motor is—

$$e i_1 = (V - i_1 R_a) i_1,$$

where V volts is the p. d. applied to the brushes and R_a the resistance of the armature circuit.

If the armature revolves at n revolutions per second, the turning moment or *torque* T is given by—

$$T 2\pi n = e i_1.$$

If T is in dyne-cms.—

$$T2\pi n = ei_1 10^7;$$

or

$$T = \frac{ei_1 10^7}{2\pi n} \text{ dyne-cms.}$$

If T is in ft.-lbs.—

$$T = 0{\cdot}118 \frac{ei_1}{n} \text{ ft.-lbs.}$$

Otherwise, remembering that 746 watts equal 1 horse-power—that is, 550 ft.-lbs. per second—

$$ei_1 \text{ watts}$$

$$= \frac{ei_1}{746} \times 550 \text{ ft.-lbs. per second.}$$

Hence

$$T = \frac{ei_1}{2\pi n} \frac{550}{746}$$

$$= 0{\cdot}118 \frac{ei_1}{n} \text{ ft.-lbs.}$$

As an example, suppose the length of the conductor in the magnetic field in Fig. 251 is 50 cms., carries a current of 100 ampères, and is placed in a field of intensity, H=10000 c.g.s. lines per sq. cm. The force on the conductor will be—

$$F = 10000 \times \frac{100}{10} \times 50 \text{ dynes}$$

$$= 5{\cdot}0 \times 10^6 \text{ dynes}$$

$$= 11{\cdot}2 \text{ lbs. weight.}$$

If the conductor be fixed to the surface of a cylindrical core of 2 ft. diameter, the torque on the core will be 11·2 ft.-lbs.

201. Coefficient of Self-Induction.—As has been stated in § 180, a current in a conductor involves the existence of a magnetic field in the neighbourhood of the conductor. Now consider a coil of a single turn of wire, and let N be the flux of magnetic induction in c.g.s. lines which links the coil when a current of C electro-magnetic units flows in the coil. If there is no magnetic substance (*e.g.*, iron) in the neighbourhood, the flux will be proportional to the current, that is—

$$N = KC,$$

where K is a constant.

If the current is i ampères—

$$N = K \frac{i}{10} \text{ (see § 185).}$$

Now suppose the flux N to vary. An e. m. f. e will be induced in the coil such that—

$$e = -\frac{1}{10^8}\frac{dN}{dt} \text{ volts}$$

$$= -\frac{1}{10^9} K \frac{di}{dt} \text{ volts}$$

$$= -L\frac{di}{dt} \text{ volts,}$$

where $$L = \frac{K}{10^9}.$$

If $\frac{di}{dt}$ is unity, the magnitude of the induced e. m. f. is $= L$. The constant L is termed the **coefficient of self-induction** or the **inductance** of the coil, measured in **henrys.** Hence the coefficient of self-induction of the coil, measured in henrys, is equal to the magnitude of the induced e. m. f. in volts induced in the coil when the current varies at the rate of 1 ampère per second.

The constant K is the coefficient of self-induction of the coil measured in c.g.s. units.

If i is 1 ampère, $K = 10N$ and $L = \frac{N}{10^8}$. Hence the coefficient of self-induction of the coil, measured in henrys, is the flux of magnetic induction linking the coil when 1 ampère flows in the coil, divided by 10^8.

Suppose now that the coil has w turns concentrated so that when a current of C electro-magnetic units flows in the coil the flux N links the whole of the w turns. If the flux vary, an e. m. f. $\frac{1}{10^8}\frac{dN}{dt}$ volts will be induced *in each turn* of the coil, and since the w turns are in series, the total e. m. f. induced in the coil will be—

$$e = -\frac{w}{10^8}\frac{dN}{dt} \text{ volts.}$$

Now the flux linking the coil will be—

$$N = KwC;$$

hence $$e = -\frac{w}{10^8} Kw \frac{dC}{dt} \text{ volts}$$

$$= -w^2 \frac{K}{10^9}\frac{di}{dt},$$

where i is the current in ampères;

or $$e = -L\frac{di}{dt} \text{ volts},$$

where $$L = \frac{Kw^2}{10^9} \text{ henrys}.$$

That is to say, the coefficient of self-induction is directly proportional to the *square* of the number of turns in the coil.

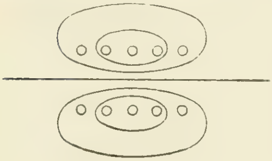


FIG. 253.

Now suppose a coil is wound, as in Fig. 253, so that the turns are not concentrated. If a current flows in the coil the magnetic flux produced will not all link every turn of the coil—the turns near the middle of the coil being linked with more flux than the turns near the end. Let the coil have w turns, and suppose a current of C electro-magnetic units flows in the coil so that a flux of N c.g.s. lines links the whole of the w turns, N_1 lines links only w_1 turns, N_2 lines links w_2 turns, and so on. The e. m. f. induced in the coil when the current varies will be—

$$e = -\left(w\frac{dN}{dt} + w_1\frac{dN_1}{dt} + w_2\frac{dN_2}{dt} + \ . \ . \ . \right)$$

$$= -\frac{1}{10^8}\frac{d}{dt}(wN + w_1N_1 + w_2N_2 + \ . \ . \ .)$$

$$= -\frac{1}{10^8}\frac{d}{dt}\Sigma w_xN_x,$$

where N_x represents the flux which links w_x turns. For a given coil Σw_xN_x will be proportional to C—that is,

$$e = -\frac{1}{10^8}\frac{d}{dt}(K_1C), \text{ where } K_1 \text{ is a constant}$$

$$= -\frac{1}{10^8}\frac{d}{dt}\left(K_1\frac{i}{10}\right), \text{ where } i \text{ is the current in ampères}$$

$$= -L\frac{di}{dt} \text{ volts.}$$

K_1 is the flux linkages for 1 electro-magnetic unit of current.

$\frac{K_1}{10}$ is the flux linkages for 1 ampère.

$$L = \frac{K_1}{10^9} = \frac{\text{flux linkages for 1 ampère.}}{10^8}$$

The coefficient of self-induction of a coil may be defined in two equivalent ways—viz.:

> **I. The coefficient of self-induction, in henrys, of a coil is the e. m. f. in volts induced in the coil when the current varies at the rate of 1 ampère per second.**
>
> **II. The coefficient of self-induction of a coil measured in henrys is equal to the number of flux linkages of the coil, divided by 10^8, when a current of 1 ampère flows in the coil.**

These two definitions are alternative statements of the same result.

If there is iron in the circuit of the magnetic flux, the coefficient of self-induction is not a constant quantity, but depends on the flux density of the iron—that is, on the magnitude of the current in the coil (see § 115).

If a current of i ampères produces a flux linkage X, the coefficient of self-induction at the flux density in the magnetic circuit corresponding to i ampères in the coil is—

$$\frac{X}{i10^8} \text{ henrys.}$$

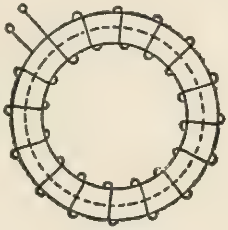


FIG. 254.

Example.—Suppose a ring of iron is uniformly wound with an exciting coil, as shown in Fig. 254.

Let the cross-sectional area of the ring be A=6 sq. cms., the mean length l=20 cms., and the number of turns in the exciting coil w=190.

Then $$\frac{4\pi}{10}\, iw = \mathrm{H}l \text{ (see p. 264),}$$

or $$\mathrm{H} = \frac{1{\cdot}257 \times 190}{20}\, i = 12i.$$

If $i=\frac{1}{2}$ ampère: H=6: B=12600 (see Fig. 131).

$$\mathrm{L} = \frac{\mathrm{BA}w}{i10^8} = \frac{12600 \times 6 \times 190}{\frac{1}{2}10^8} = 0{\cdot}288 \text{ henry.}$$

If i=1 ampere: H=12: B=14200:

$$\mathrm{L} = 0{\cdot}162 \text{ henry.}$$

If i=2 ampères: H=24: B=15600:

$$\mathrm{L} = 0{\cdot}089 \text{ henry.}$$

202. The Energy of the Magnetic Field.—It has been explained in § 201 that when a current of i ampères flows in a coil of self-induction, L henrys, the flux linkages of the coil will be—

$$10^8\mathrm{L}i.$$

Further, the work done in causing a current of i ampères to link with a flux N c.g.s. lines is (§ 188)—

$$\frac{Ni}{10} \text{ ergs.}$$

In the case of a coil of self-induction L henrys the work done in increasing the current from i to $\delta i + i$ ampères is intermediate between—

$$\frac{\delta S i}{10} \text{ ergs and } \frac{\delta S\ (i+\delta i)}{10} \text{ ergs,}$$

where S is the total flux linkage when the current i ampères flows in the coil—that is,

$$S = \Sigma w_x N_x;$$

or, in the notation of the calculus, when the increment are indefinitely small, the work done is—

$$\frac{idS}{10} \text{ ergs, where } dS = 10^8 d\ (Li).$$

Hence the work done is—

$$10^7 Lidi \text{ ergs} = Lidi \text{ joules,}$$

since 1 joule $= 10^7$ ergs (see p. 201).

The total work done in establishing the current of i ampères in a coil of self-induction of L henrys is—

$$\int_o^i Lidi \text{ joules}$$

$$= \tfrac{1}{2} Li^2 \text{ joules.}$$

The same result may be arrived at in a somewhat different manner, as follows:

The back e. m. f. induced in the coil when the current increases from i to $i + \delta i$ is—

$$e = -L\frac{di}{dt} \text{ volts.}$$

The energy supplied in time δt in driving the current i *against* this back e. m. f. is (see § 145)—

$$vi = -ei\delta t \text{ joules} = Li\frac{di}{dt}\delta t.$$

The total energy supplied in establishing the current is therefore—

$$\int^i Lidi \text{ joules}$$

$$= \tfrac{1}{2} Li^2 \text{ joules.}$$

The above results hold if L is constant throughout the range of flux considered—that is, if the flux is proportional to the current throughout the range of current considered.

In order to calculate the energy of the magnetic field, however, when there is iron in the magnetic circuit, it is necessary to know the relationship between the flux and the current. For example, in the case of an iron ring magnetised by means of a closely and uniformly wound exciting coil, as shown in Fig. 254, suppose the curve connecting the values of B and H be as shown in Fig. 255. The energy expended in producing the flux in the ring is—

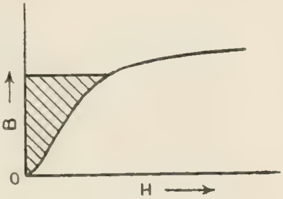


FIG. 255.

$$-\int eidt \text{ joules,}$$

where e is the back e. m. f. developed by the changing flux, i is the corresponding current in the coil, and the integration is taken over the time taken in establishing the flux in the ring. Hence the energy is—

$$\frac{1}{10^8}\int\frac{d(\mathrm{BA}w)}{dt}\,idt \text{ joules,}$$

since $$e = -\frac{d}{dt}\left(\frac{\mathrm{BA}w}{10^8}\right) \text{ volts (see § 197),}$$

where w is the number of turns in the exciting coil.

The energy expended in producing the flux is therefore—

$$= \frac{1}{4\pi\times 10^7}\int_0^{\mathrm{B}_1}\mathrm{H}l\mathrm{A}d\mathrm{B} \text{ joules, since } \frac{4}{10}iw = \mathrm{H}l \text{ (p 264),}$$

$$= \frac{1}{4\pi}\int_0^{\mathrm{B}_1}\mathrm{H}d\mathrm{B} \text{ ergs per c.cm.;}$$

that is, the energy expended in ergs per c.cm. of the iron, in producing a flux density B, is equal to the shaded area in Fig. 255 divided by 4π (see § 116, Fig. 143).

Since 1 erg is equal to $0{\cdot}737\times 10^{-7}$ ft.-lbs., the energy expended is equal to the shaded area multiplied by $\frac{0{\cdot}737}{4\pi\times 10^7}$, in ft.-lbs. per c.cm.

If the iron is not uniformly magnetised, the energy necessary to magnetise any element to an induction density B_1 is—

$$\frac{\delta \mathrm{V}}{4\pi}\int \mathrm{H}d\mathrm{B} \text{ ergs,}$$

where δV is the volume of the element in cubic cms. (see also §§ 116 and 117, Chapter VIII.).

If the current in a solenoid is suddenly broken by opening the switch, the energy of the magnetic field is dissipated in the spark at the switch contacts. Further, if the circuit is very quickly broken so that the current is forced to cease in a very short time, the e. m. f. induced in the coil due to the stoppage of the current may be sufficiently large to rupture the insulation. Thus, since the *magnitude* of the e. m. f. induced is—

$$L \frac{di}{dt} \text{ volts,}$$

it follows that if the current is stopped very quickly the magnitude of $\frac{di}{dt}$ will be very great, and consequently the induced e. m. f. e also will be very large (see Fig. 237).

It is frequently the practice to connect in parallel with an inductive coil a non-inductive resistance, as shown in Fig. 256, so that when the switch is opened the current in the inductive coil may have a path through the non-inductive resistance and thus be allowed to die away slowly.

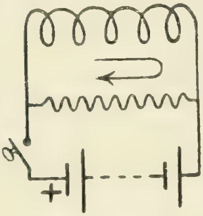


FIG. 256.

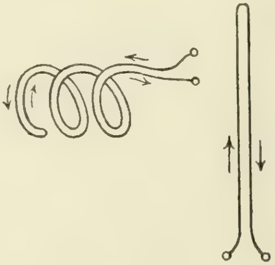


FIG. 257.

If it is desired to wind a coil so that its coefficient of self-induction may be as small as possible, the wire may be doubled closely back on itself and then wound as a coil (Fig. 257). In this way the fields due to the current in each half of the wire mutually neutralise each other to a very large extent, so that the flux linking the coil becomes relatively very small.

203. Energy Loss in Iron due to Hysteresis.—In § 116 an expression was deduced for the energy loss in iron, due to hysteresis, from an application of the laws of magnetism. The same result may be deduced from the law of electro-magnetic induction.

For example, suppose the ring shown in Fig. 258 is magnetised by passing a current through the exciting coil, and suppose the current be taken through a complete cycle so that the B-H loop, as shown in Fig. 259, is obtained. From the results of p. 287 it follows that the energy loss in performing the cycle is—

$$U=\frac{1}{4\pi}\int_{0} H\,dB \text{ ergs per c.cm.,}$$

the integration being performed over the complete cycle, and, as in § 116, Chapter VIII., this integration is the area of the

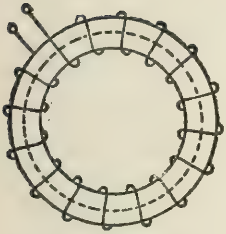


Fig. 258.

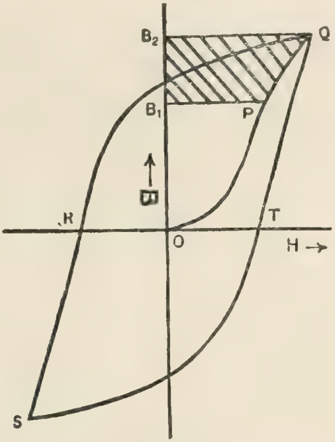


Fig. 259.

B–H loop. Hence the energy loss in hysteresis in the iron in performing a complete cycle of B–H changes is—

$$\left(\frac{1}{4\pi} \text{ area of the B–H loop}\right) \text{ ergs per c.cm.}$$

204. Growth of Current in a Coil of Inductance L Henrys and Resistance R Ohms.—Suppose the coil be connected to a source of steady e. m. f. of E volts. A current will begin to flow in the coil and a corresponding magnetic flux will be developed linking the coil. By Lenz's Law the increasing flux will induce a back e. m. f. $L\frac{di}{dt}$ volts, in the direction opposing the growth of the flux—that is, opposing the rise of the current. The actual effective e. m. f. in the circuit at any instant is therefore—

$$E-L\frac{di}{dt} \text{ volts,}$$

and this is the e. m. f which is available for driving the current i through the coil of resistance R ohms.

Hence $$E-L\frac{di}{dt}=Ri$$

From this equation the value of the current may be deduced at any instant after closing the switch.

Writing the equation thus—

$$E = L\frac{di}{dt} + Ri,$$

and noting that at the instant of closing the switch the current i is zero, it follows that—

$$\left.\frac{di}{dt}\right]_{t=o} = \frac{E}{L},$$

so that a line drawn through the origin in Fig. 260, and at such an inclination θ_1 to the axis of t that—

$$\tan\theta_1 = \frac{E}{L}$$

gives the initial slope of the required curve. After a small interval δt, the current will be $i_1 = \tan\theta_1\delta t$, and hence at this instant t_1—

$$E = L\left.\frac{di}{dt}\right]_{t=t_1} + Ri_1;$$

or

$$\left.\frac{di}{dt}\right]_{t=t_1} = \frac{E - Ri_1}{L} = \tan\theta_2,$$

and this is the slope of the curve at the instant $t = t_1$.

In this way the whole curve showing the rise of the current may be plotted, and by taking the time intervals δt between successive points sufficiently small the result may be made as accurate as is desired.

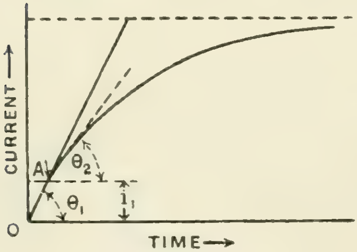


FIG. 260.

This method of graphically deducing the shape of the curve of rise of current is very useful in special cases, but in the present case in which E, L, and R are assumed to be constants throughout the process, a much simpler method is to write down the solution of the differential equation—

$$L\frac{di}{dt} + Ri = E.$$

This is a well-known result—viz.,

$$i = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

or

$$i = I\left(1 - e^{-\frac{R}{L}t}\right),$$

where I is the final steady value which the current eventually reaches, and $e = 2{\cdot}718$ is the base of natural logarithms.

Differentiating i with respect to t gives—

$$\frac{di}{dt} = \mathrm{I}\,\frac{\mathrm{R}}{\mathrm{L}}\,e^{-\frac{\mathrm{R}}{\mathrm{L}}t}$$

and hence, at the moment of closing the switch—that is, for $t = o$—

$$\left.\frac{di}{dt}\right]_{t=o} = \frac{\mathrm{I}}{\frac{\mathrm{L}}{\mathrm{R}}} = \frac{\mathrm{E}}{\mathrm{L}}.$$

The quotient $\frac{\mathrm{L}}{\mathrm{R}}$ is termed the *time constant* of the coil, and is the time in seconds in which the current would reach its final value I if the initial rate of increase of i were maintained constant and equal to $\frac{\mathrm{E}}{\mathrm{L}}$.

The time constant $\frac{\mathrm{L}}{\mathrm{R}}$ is the factor which determines the rate of rise of current in the coil.

Thus, in Fig. 261—

$$\left.\frac{di}{dt}\right]_{t=o} = \tan\theta = \frac{\mathrm{I}}{\frac{\mathrm{L}}{\mathrm{R}}} = \frac{\mathrm{E}}{\mathrm{L}}.$$

Actually in the time $t = \frac{\mathrm{L}}{\mathrm{R}}$, the current reaches the value—

$$i = \mathrm{I}\,(1 - e^{-1}) = \mathrm{I}\left(1 - \frac{1}{e}\right) = \mathrm{I}\,(1 - 0{\cdot}368) = 0{\cdot}632\ \mathrm{I}.$$

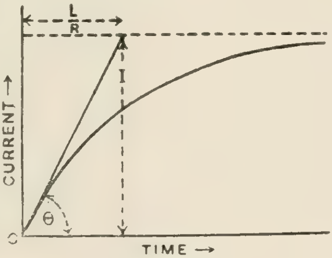


Fig. 261.

Hence the time constant is the time taken for the current to reach a value $= 63{\cdot}2$ per cent. of its final value.

Strictly speaking, the current never reaches a final steady value, but goes on increasing indefinitely, as will be evident from the equation deduced for the current curve. The time in which the current reaches within any desired percentage of the final value is easily deduced as follows—viz., when the current has reached 98 per cent. of its final value—

$$0{\cdot}98\mathrm{I} = \mathrm{I}\left(1 - e^{-\frac{\mathrm{R}}{\mathrm{L}}t}\right)$$

$$0{\cdot}02 = e^{\frac{\mathrm{R}}{\mathrm{L}}t}$$

$$\log_e 0{\cdot}02 = -\frac{\mathrm{R}}{\mathrm{L}}\,t,$$

or
$$t=-\frac{\log_e 0{\cdot}02}{\frac{R}{L}}=\frac{3{\cdot}9}{\frac{R}{L}}.$$

If $\frac{R}{L}=1: t=3{\cdot}9$ seconds.

If $\frac{R}{L}=100: t=0{\cdot}039$ second.

In Fig. 262 is shown the rise of current in a coil of which the constants are as follows:

$$E=200 \text{ volts}: L=0{\cdot}4 \text{ henry}: R=10 \text{ ohms}.$$

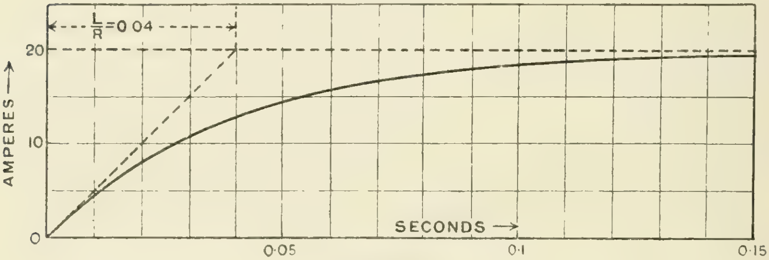


Fig. 262.

205. Quantity of Electricity set in Motion in Electro-Magnetic Induction.—Consider a closed circuit (Fig. 238) of resistance R ohms and linked by a steady flux of N c.g.s. lines of magnetic induction. If this flux be withdrawn from the coil (or if the coil be removed from the neighbourhood of the flux) an e. m. f. e will be induced such that—

$$e=-\frac{1}{10^8}\frac{dN}{dt} \text{ volts,}$$

and a current $i=\frac{e}{R}$ ampères will flow. A quantity of electricity of q coulombs will therefore pass round the circuit, such that—

$$q=\int i dt=\frac{1}{R}\int e dt=-\frac{1}{10^8 R}\int_N^o dN \text{ coulombs;}$$

or
$$q=\frac{N}{10^8 R} \text{ coulombs,}$$

where N is the total change in the flux linking the coil.

The quantity q is therefore dependent only on the change of flux and the resistance of the circuit, and not on the time taken to produce the change of flux.

An application of this result will be found in § 214.

206. Mutual Induction.—Suppose two coils (Fig. 263) are placed in the neighbourhood of each other, so that when a current is passed through the larger coil the flux of magnetic induction due to this current links the second coil. There is then said to be mutual induction between the two coils.

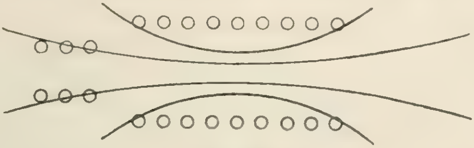


Fig. 263.

Consider an elementary tube of flux linking w_{1x} turns of coil 1 and w_{2x} turns of the coil 2. If a current of 1 ampère flows in coil 1 the elementary tube will contain the flux—

$$\frac{4\pi}{10}\times\frac{w_{1x}}{R_x},$$

where R_x is the magnetic reluctance of the tube (see p. 266).

This flux links w_{2x} turns of coil 2—that is, the flux linkages of coil 2 will be—

$$\frac{4\pi}{10}\times\frac{w_{1x}\,w_{2x}}{R_x}.$$

The total flux linkages of coil 2 due to a current of 1 ampère in coil 1 will be—

$$\frac{4\pi}{10}\Sigma\frac{w_{1x}\,w_{2x}}{R_x},$$

the summation being taken for the whole of the flux which links both coils.

This expression is termed the **coefficient of mutual induction** of coil 2 with respect to coil 1.

The coefficient of mutual induction is therefore—

$$M=\frac{\frac{4\pi}{10}\Sigma\frac{w_{1x}\,w_{2x}}{R_x}}{10^8}\text{ henrys.}$$

Again, consider the path of the elementary tube of flux which links w_{1x} turns of coil 1 and w_{2x} turns of coil 2 when a current of 1 ampère flows in coil 1. Suppose a current of 1 ampère now flows in the second coil and no current in coil 1. The flux produced in the path of reluctance R_x is—

$$\frac{4\pi}{10}\times\frac{w_{2x}}{R_x}.$$

This flux links w_{1x} turns of coil 1, and hence the flux linkages of coil 1 due to this element of flux when a current of 1 ampère flows in coil 2 is—

$$\frac{4\pi}{10}\times\frac{w_{2x}\,w_{1x}}{R_r}.$$

The total flux linkages of coil 1 due to 1 ampère in coil 2 is therefore—

$$\frac{4\pi}{10}\sum\frac{w_{2x}\,w_{1x}}{R_x},$$

taken for the whole flux which links both coils. This is the coefficient of mutual induction of coil 1 with respect to coil 2, and is the same result as that deduced for the coefficient of mutual induction of coil 2 with respect to coil 1 (see also § 106).

Hence, generally stated, the coefficient of mutual induction of two coils is the total number of flux linkages of one coil due to a current of 1 ampère in the other. Expressed in henrys—

$$M=\frac{\text{flux linkages of coil B when 1 ampère flows in coil A}}{10^8}\text{ henrys.}$$

Also—

$$M=\frac{\text{flux linkages of coil A when 1 ampère flows in coil B}}{10^8}\text{ henrys.}$$

For a further consideration of mutual induction, see § 215, also Chapter XXII., Section V.

207. Electro-Magnetic Repulsion.—In Fig. 264 is shown a solenoid with an iron core, the latter being preferably a bundle of iron wires. An aluminium ring is placed over the projecting end of the iron core.

The solenoid is connected through a switch to a battery, and it will be found that at the moment of closing the switch the aluminium ring will be shot up into the air. This is an example of electromagnetic repulsion, and is explained as follows:

When the switch is closed the current in the solenoid winding—and consequently the flux in the core—grows according to a logarithmic curve, as shown in Fig. 261. This increasing flux induces a current in the aluminium ring, the direction of the current being such that it opposes the rising flux in the solenoid core (see § 195, Lenz's Law). A force therefore acts on the ring, tending to drive it off the iron core of the solenoid, and thus to prevent it being linked by the rising flux in the iron core.

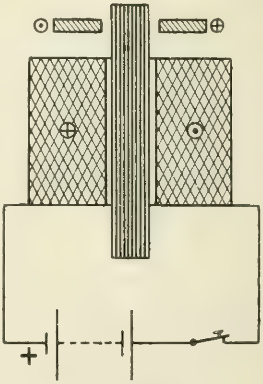


FIG. 264.

CHAPTER XVI

GALVANOMETERS — FLUXMETER — OSCILLOGRAPH — WATTMETERS — AMMETERS — VOLTMETERS — FREQUENCY METERS.

208. Thomson's Galvanometer.—Galvanometers are used for measuring currents of very small magnitude. One of the earliest and most successful types of sensitive galvanometer which is still largely used is the Thomson galvanometer. The principle of action of this instrument is illustrated in Fig. 265. Two small magnets, secured to a supporting stem, are arranged to form as nearly an astatic combination as possible (see § 94), so that the effect of the earth's magnetic field on the combination is reduced to a minimum. Each needle is embraced by one of two coils which are connected in series, so that the current flows in opposite directions in the two coils. For example, if the current in the upper coil produces a field through the coil in a direction away from the observer, the current in the lower coil will produce a field through the coil in a direction towards the observer. Under these conditions the upper needle will tend to turn with its N pole away from the observer, and the lower needle tends to turn with its N pole towards the observer. In other words, the turning moments on the magnetic system due to the two coils are in the same direction.

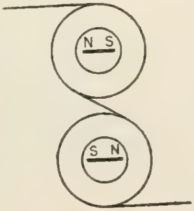


Fig. 265.

In Fig. 266 is shown the complete instrument as manufactured by the Cambridge and Paul Instrument Company, the magnet system and suspension being also shown separately in this figure. Two groups, each containing 4 magnets, are suspended by a fine quartz fibre, the two groups being arranged to form an astatic system. The control of the needle—that is, the period of swing—is effected by means of two adjustable permanent magnets. In this way the time of swing of the magnet system may be varied from about 5 seconds to about 20 seconds.

The deflection is measured by means of a spot of light reflected from the mirror B on to a distant scale. The vibration of the moving system is damped by means of a vane D moving in an air chamber between adjustable plates, so that the amount of damping may be varied to suit the particular measurement for which the galvanometer is used.

The following data refer to one of these instruments:

Resistance of the two coils in series = 1965 ohms.

Time of one complete oscillation = 14·4 seconds.

Current necessary to produce a deflection of 1 mm. on a scale distant 1 metre = 0·000146 micro-ampère.

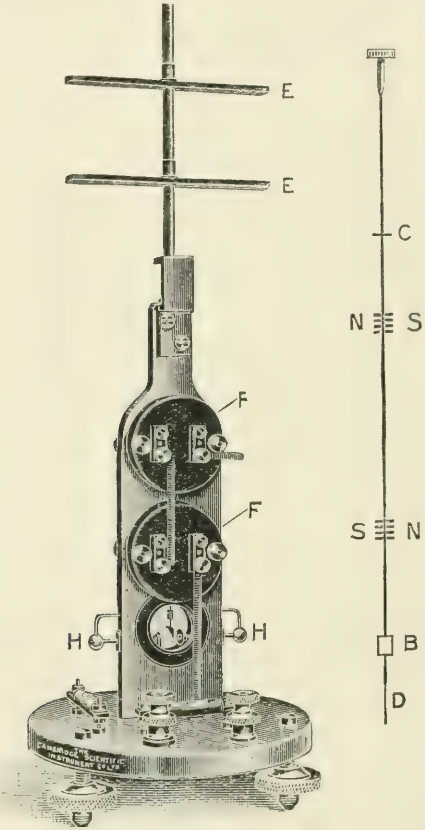


Fig. 266.

209. The Broca Galvanometer.—The special feature of this instrument is the use of vertical magnets to form an astatic system, as shown in Fig. 267. The magnets consist of two steel wires placed vertically, each being magnetised so that its two ends are of like polarity and having intermediate (*i.e.*, consequent) poles near the middle. The current in the coil produces a magnetic field which is in one direction in the neighbourhood of the consequent poles, and in the opposite direction in the neighbourhood of the ends of the magnets. The turning moment due to the action of the field on the consequent poles is thus in the same sense as that due to the action of the field on the poles at the ends of the magnets.

The vertical arrangement of the magnets makes it possible to use comparatively powerful magnets and at the same time to keep the moment of inertia of the moving system reasonably small. This is an advantage in that it allows the galvanometer to have a reasonably short period, thus allowing readings to be quickly taken even when the controlling force is feeble—that is, when the galvanometer is most sensitive.

The system is suspended by a fine quartz fibre, and the controlling force is chiefly provided by the adjustable magnet B. By adjustment of this magnet the time of swing of the moving system may be varied from about 5 seconds to about 20 seconds.

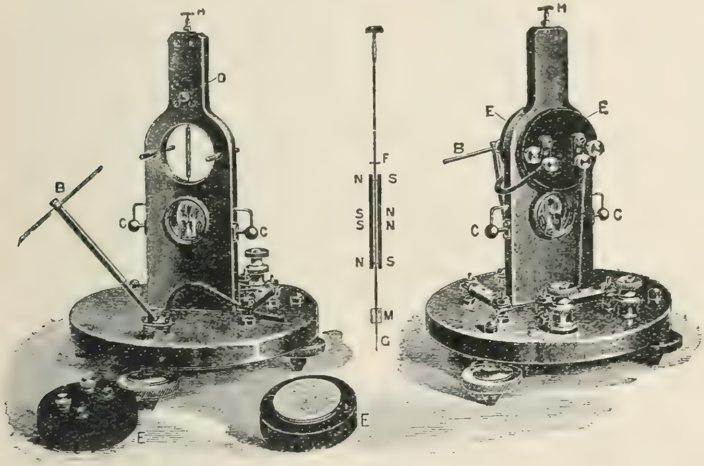


Fig. 267.

In a particular galvanometer of which the coil had a resistance of 860 ohms, a current of 0·00015 micro-ampère produced a deflection of 1 mm. on the scale at a distance of 1 metre, the time of one complete oscillation being 17·3 seconds.

210. Moving Coil Galvanometer.—In many cases the galvanometer coil is made movable, and the magnet system is fixed, such an instrument being known as a Deprez-d'Arsonval galvanometer. In Fig. 268 is shown an example of this type of instrument, a diagrammatical plan view of the magnet and coil being shown in Fig. 269. The rectangular coil is movable about a vertical axis formed by two wires by which the current respectively enters and leaves the coil. A horse-shoe permanent magnet provides the magnetic field, and a cylinder of soft iron arranged inside the coil forms a low reluctance path for the magnetic flux between the poles. When a current is passed through the coil a force is produced tending to set the coil with its plane at right angles to the direction of the magnetic field, the deflection being opposed by the torsion of the suspension wire.

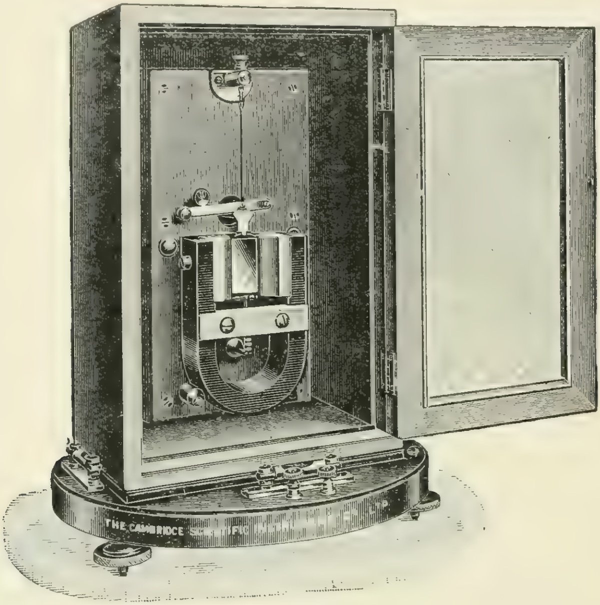


Fig. 268.*

The position assumed by the coil is such that these two opposing forces balance. In order to damp out vibrations of the coil so that it may quickly assume a steady deflection, it may be wound on a light silver frame, in which case the movement in the magnetic field induces currents in the frame which quickly damp out the vibrations. The deflection is read by means of a spot of light reflected from a mirror secured to the suspension of the coil and intercepted by a distant scale.

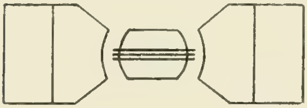


Fig. 269.

The magnetic field due to the permanent magnet is very strong in the neighbourhood of the coil, so that external fields such as that due to the earth will have no appreciable disturbing effect on the deflections.

211. Einthoven "String" Galvanometer.—In this instrument the moving "coil" is reduced to a single wire or "string," which in the most sensitive instruments consists of a very fine glass fibre (about 0·002 mm. diameter), coated with silver. The string is stretched in a very narrow air gap between the poles of a powerful electro-magnet. When a current passes along the string a deflection is produced, as shown in Fig. 270. The movement is observed by means of a microscope A passing through a hole bored in the pole shoe, or, alternatively, an enlarged image of the wire may be thrown on to a screen by means of an arc lamp and projection lantern.

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* The constructional details are somewhat different in the latest pattern.

The moment of inertia of the moving system is extremely small. By adjusting the tension on the string the time of vibration may be reduced to less than $\frac{1}{200}$ second. If the tension is reduced the string may take several seconds to reach its final deflection. If the tension on the string is not too great the instrument is " aperiodic " or " dead-beat "—that is, it attains its full deflection without vibration.

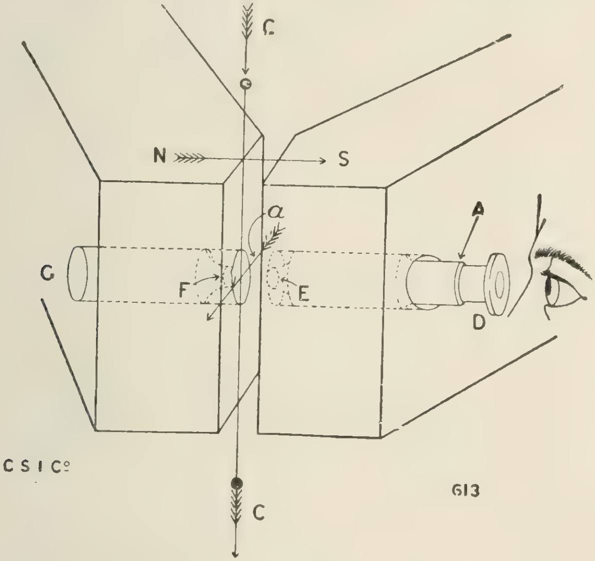

Fig. 270.

Used in conjunction with a photographic recording apparatus, the instrument may be used to record alternating or pulsating currents of very small magnitude, such as those received in wireless telegraphy and telephony.

The following data supplied by the Cambridge and Paul Instrument Company show the degree of sensitivity of which this galvanometer is capable.

For a silvered glass " string," 0·002 mm. diameter and resistance 6610 ohms, a deflection of 1 mm. was produced by a current of 0·000015 micro-ampère.

212. Ballistic Galvanometer.—When a condenser is discharged through a galvanometer, the moving system of which is undamped, a deflection is produced, and the moving system comes to rest after a series of oscillations. If the discharge is complete before an appreciable deflection is produced, the angle of throw of the moving system measures the *quantity* of electricity passed through the galvanometer by the discharge.

The conditions which a galvanometer, suitable for measuring electrical quantity, should fulfil are: (i.) the moving system should have a large moment of inertia so that the discharge through the galvanometer may be complete before the moving system is appreciably deflected; and (ii.) the damping of the moving system should be small so that, other conditions being the same, the "throw," or first full deflection, of the galvanometer may be as large as possible. Thus, if the galvanometer be of the type shown in Fig. 268—if the coil be wound on a metal frame—this frame should be split along a line perpendicular to the axis of the coil, so that the electro-magnetic damping effect may be made small. The importance of this condition of small damping effect will be apparent from the consideration of the theory of the ballistic galvanometer given in the next paragraph.

213. Theory of the Ballistic Galvanometer.—

Let K be the moment of inertia of the moving system.
,, θ ,, deflection at any instant.
,, $\frac{a\,d\theta}{dt}$,, damping couple when the velocity is $\frac{d\theta}{dt}$.
,, $b\theta$,, controlling couple tending to cause the moving system to return to the position of equilibrium (that is, of zero deflection).

In the case of a moving coil instrument the controlling force will be the torsion on the suspension wire. In the case of a moving magnet instrument the controlling force will be largely due to the control magnet. In this case the controlling force will be proportional to $\sin\theta$, but usually θ is sufficiently small to assume that $\sin\theta=\theta$ (see p. 305).

When the discharge through the galvanometer is complete, the moving system starts off with a certain velocity dependent on the quantity of electricity passed through the galvanometer.

Case I. Moving Coil Instrument.—If i ampères is the current at any instant the deflecting couple will be ki where k is a constant of the instrument.

Hence
$$ki=\mathrm{K}\,\frac{d^2\theta}{dt^2},$$

or
$$\int ki\,dt=\int \mathrm{K}\,\frac{d^2\theta}{dt^2};$$

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that is—

$$k\int i\,dt=\mathrm{K}\frac{d\theta}{dt}.$$

or

$$k\mathrm{Q}=\mathrm{K}\frac{d\theta}{dt}.$$

Hence the initial angular velocity of the coil will be—

$$\omega=\frac{d\theta}{dt}=\frac{k}{\mathrm{K}}\mathrm{Q}$$

—that is, directly proportional to the quantity of electricity Q coulombs which is passed through the instrument.

If the moving system is a magnet system a similar argument holds (see p. 304).

The moving coil then starts with an initial velocity—

$$\omega=\frac{d\theta}{dt}=\frac{k}{\mathrm{K}}\mathrm{Q},$$

and will continue oscillating until the energy is dissipated by friction or electro-magnetic damping. The equation of motion is then as follows:

$$\mathrm{K}\frac{d^2\theta}{dt^2}+a\frac{d\theta}{dt}+b\,\theta=o;$$

or

$$\frac{d^2\theta}{dt^2}+\frac{a}{\mathrm{K}}\frac{d\theta}{dt}+\frac{b}{\mathrm{K}}\theta=o.$$

If the *damping factor a is relatively small* the solution of this equation is (see also § 278)—

$$\theta=\mathrm{B}e^{-\frac{1}{2}\frac{a}{\mathrm{K}}t}\cos\left[\sqrt{\frac{b}{\mathrm{K}}}\,t+\beta\right]\text{ radians,}$$

where B and β are constants.

But $\theta=o$, and $\frac{d\theta}{dt}=\omega$, when $t=o$.

Hence

$$\theta=\omega\sqrt{\frac{\mathrm{K}}{b}}\,e^{-\frac{1}{2}\frac{a}{\mathrm{K}}t}\sin\sqrt{\frac{b}{\mathrm{K}}}\,t\text{ radians.}$$

The time for the first full deflection is—

$$t_1=\frac{\pi}{2}\sqrt{\frac{\mathrm{K}}{b}}\text{ seconds,}$$

and the amplitude of the first swing (Fig. 271) is—

$$\theta_1=\omega\sqrt{\frac{\mathrm{K}}{b}}\,e^{-\frac{1}{2}\frac{a}{\sqrt{\mathrm{K}b}}\frac{\pi}{2}}$$

For the next full deflection on the opposite side (see Fig. 271)—

$$t_2 = \frac{3\pi}{2}\sqrt{\frac{K}{b}} \text{ seconds,}$$

and the amplitude of this second swing will be—

$$\theta_2 = \omega\sqrt{\frac{K}{b}}\, e^{-\frac{1}{2}\frac{a}{\sqrt{Kb}}\frac{3\pi}{2}} \text{ radians.}$$

Similarly $$\theta_3 = \omega\sqrt{\frac{K}{b}}\, e^{-\frac{1}{2}\frac{a}{\sqrt{Kb}}\frac{5\pi}{2}};$$

and $$\theta_n = \omega\sqrt{\frac{K}{b}}\, e^{-\frac{1}{2}\frac{a}{\sqrt{Kb}}(2n-1)\frac{\pi}{2}}.$$

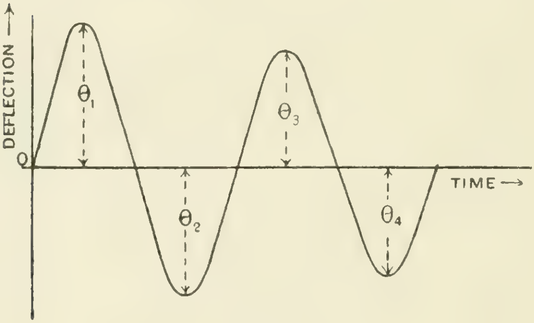


Fig. 271.

If there had been no damping a would be zero, and the amplitude of each swing would be—

$$\Theta = \omega\sqrt{\frac{K}{b}} \text{ radians.}$$

Hence actually $$\Theta = \theta_1 e^{\frac{1}{2}\frac{a}{\sqrt{Kb}}\frac{\pi}{2}} \text{ radians.}$$

But from the results given above—

$$\frac{\theta_1}{\theta_2} = e^{\frac{1}{2}\frac{a}{\sqrt{Kb}}\pi};$$

hence $$\Theta = \theta_1\sqrt{\frac{\theta_1}{\theta_2}} \text{ radians}$$

Otherwise $$\log_e\frac{\theta_1}{\theta_2} = \frac{1}{2}\frac{a}{\sqrt{Kb}}\pi = \lambda.$$

The quantity λ is termed the logarithmic decrement,

and $$\Theta=\theta_1 e^{\frac{\lambda}{2}};$$

or $$\Theta=\theta_1\left(1+\frac{\lambda}{2}\right) \text{ very approximately,}$$

since for small values of λ—

$$e^{\frac{\lambda}{2}}=\left(1+\frac{\lambda}{2}\right) \text{ very approximately (see § 282).}$$

It will be seen that the following relationships also hold—

$$\frac{\theta_1}{\theta_2}=\frac{\theta_2}{\theta_3}=\frac{\theta_3}{\theta_4}=\frac{\theta_{n-1}}{\theta_n}.$$

In determining λ it is usually more accurate to note the 1st swing and the nth swing (see Fig. 271).

Then from the results previously obtained—

$$\theta_n=\Theta e^{-\frac{\lambda}{2}(2n-1)},$$

and $$\theta_1=\Theta e^{-\frac{\lambda}{2}},$$

hence $$\frac{\theta_n}{\theta_1}=e^{-\lambda(n-1)},$$

or $$\lambda=\frac{1}{n-1}\log_e\frac{\theta_1}{\theta_n}.$$

The value of λ may thus be deduced from the ratio of the amplitude of the 1st swing to that of the nth swing, and the undamped deflection Θ obtained as before from the equation—

$$\Theta=\theta_1\left(1+\frac{\lambda}{2}\right) \text{ radians.}$$

The value of λ will of course alter if the resistance of the galvanometer circuit is altered.

The quantity of electricity discharged through the galvanometer may be deduced from the value of Θ as follows:

$$\Theta=\omega\sqrt{\frac{K}{b}} \text{ radians,}$$

and the initial velocity ω has been shown to be—

$$\omega=\frac{k}{K}Q \text{ radians per second.}$$

Hence $$Q=\Theta\frac{\sqrt{bK}}{k} \text{ coulombs.}$$

If T is the time of one *complete* swing of the galvanometer coil (*i.e.*, a swing from one side to the next swing on the same side)—

$$T = 2\pi\sqrt{\frac{K}{b}} \text{ seconds.}$$

Hence $$Q = \frac{b}{k}\,\frac{T}{2\pi}\,\Theta \text{ coulombs.}$$

The quantity $\frac{b}{k}$ is a constant of the galvanometer which may be determined as follows:

(i) A known quantity of electricity Q_o may be passed through the galvanometer (*e.g.*, by discharging through the galvanometer a standard condenser charged to a known p. d. by means of a standard cell). If Θ_o radians be the amplitude of the undamped swing of the galvanometer—

$$\frac{b}{k} = \frac{Q_o}{\Theta_o}\cdot\frac{2\pi}{T}.$$

(ii) If a steady current of known value i ampères is passed through the galvanometer (*e.g.*, by connecting the galvanometer coil in series with a known resistance and a standard cell), and if X radians is the steady deflection thus produced, it follows from the results obtained in the early part of this discussion that—

$$ki = bX,$$

and hence $$\frac{b}{k} = \frac{i}{X}.$$

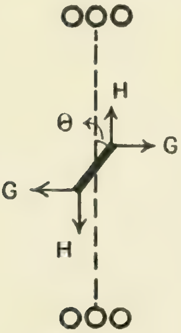


FIG. 272.

Case II. Moving Magnet Instrument.—If the galvanometer is as shown in Fig. 272, in which the plane of the coil is in the magnetic meridian and the magnet needle pivoted horizontally at the centre of the coil, then when a current i ampères is passed round the coil the couple tending to deflect the needle at right angles to the plane of the coil is—

$$iGM\cos\theta,$$

where M is the magnetic moment of the needle and G is the intensity of the field at the centre of the coil due to a current of 1 ampère in the coil (see also §§ 126 and 185).

The couple tending to restore the needle to the magnetic meridian is—

$$MH\sin\theta.$$

Hence when a steady deflection is obtained—

$$iGM\cos\theta = MH\sin\theta;$$

or $$i = \frac{H}{G}\tan\theta.$$

That is to say, the current is proportional to the tangent of the angle of deflection of the needle. For this reason the galvanometer is known as a *tangent galvanometer* (see also § 190).

If a discharge of electricity is passed through the coil and the moment of inertia K of the needle is so large that the discharge is completed before the needle has moved appreciably, then—

$$\mathrm{GM}i=\mathrm{K}\,\frac{d^2\theta}{dt^2},$$

since the deflection θ is zero during the discharge.

Hence $\mathrm{GMQ}=\int \mathrm{GM}i\,dt=\mathrm{K}\,\frac{d\theta}{dt}$; that is, $\mathrm{GMQ}=\mathrm{K}\omega$,

where Q coulombs is the quantity passed through the galvanometer and ω is the angular velocity with which the needle starts off.

The equation of motion of the needle may then be written:

$$\mathrm{K}\,\frac{d^2\theta}{dt^2}+a\,\frac{d\theta}{dt}+\mathrm{MH}\sin\theta=0.$$

For small values of θ a close approximation is obtained by writing $\sin\theta=\theta$, and the equation of motion then becomes—

$$\mathrm{K}\,\frac{d^2\theta}{dt^2}+a\,\frac{d\theta}{dt}+\mathrm{MH}\theta=0$$

If the *damping factor a is small*, the solution of this equation is as previously found—viz.:

$$\theta=\omega\sqrt{\frac{\mathrm{K}}{\mathrm{MH}}}\,e^{-\frac{1}{2}\frac{a}{\mathrm{K}}t}\sin\sqrt{\frac{\mathrm{MH}}{\mathrm{K}}}\,t;$$

that is, the undamped deflection is—

$$\Theta=\omega\sqrt{\frac{\mathrm{K}}{\mathrm{MH}}}\text{ radians}$$

But it has been shown that $\omega=\frac{\mathrm{GMQ}}{\mathrm{K}}$.

Hence $$\mathrm{Q}=\frac{\sqrt{\mathrm{KH}}}{\mathrm{G}\sqrt{\mathrm{M}}}\,\Theta;$$

and, since $$\mathrm{T}=2\pi\sqrt{\frac{\mathrm{K}}{\mathrm{MH}}}\text{ seconds,}$$

$$\mathrm{Q}=\frac{\mathrm{H}}{\mathrm{G}}\,\frac{\mathrm{T}}{2\pi}\Theta\text{ coulombs.}$$

The value of the constant $\frac{H}{G}$ may be determined by precisely the same methods as used to determine the value of $\frac{b}{k}$ in the case of the moving coil galvanometer.

Thus, when a steady current of i ampères is passed through the galvanometer and a steady deflection θ is obtained—

$$\frac{H}{G}=i\cot\theta;$$

or, for small values of θ—

$$\frac{H}{G}=\frac{i}{\theta}.$$

214. Determination of B-H Curves by means of the Ballistic Galvanometer.

214. Determination of B-H Curves by means of the Ballistic Galvanometer.—Suppose the iron sample of which the B–H curve is to be determined is prepared in the form of a closed ring, and let it be uniformly wound with a coil which may be supplied with current from a battery B (Fig. 273) through a reversing switch S_W, an ammeter A and adjustable resistances R_1 and R_2. When the switch S_W is closed, the resistance R_2 is directly connected across the battery, and the current for the exciting coil of the iron ring specimen is tapped off from the resistance R_2.

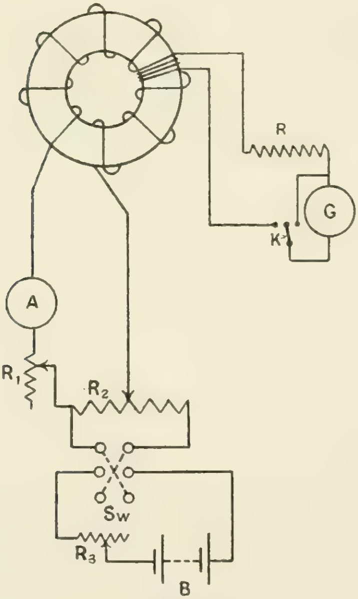


FIG. 273.

A second coil of a large number of turns of fine wire is wound over and insulated from the exciting coil, and connected through a resistance to the ballistic galvanometer G.

Suppose the primary (*i.e.*, exciting) coil has w_1 turns and the secondary coil w_2 turns. If a current i ampères is passed through the primary (the galvanometer switch K being kept open) a certain flux N will be established in the iron ring. If the galvanometer switch K is now closed and the current i is reversed by means of the switch S_W, the flux N will be established in the ring in the reverse direction, so that the total flux change in the ring will be 2N lines.

Now the flux 2N will cut the w_2 turns of the secondary winding, and if r is the total resistance of the galvanometer circuit the quantity of electricity which will be discharged through the galvanometer is by § 205—

$$\frac{2Nw_2}{r10^8} \text{ coulombs.}$$

Hence by § 213—

$$\frac{2Nw_2}{r10^8}=\frac{b}{k}\frac{T}{2\pi}\Theta,$$

where $\frac{b}{k}$ is a constant of the galvanometer, Θ is the undamped deflection, and T is the time of a complete swing.

Further $$N=BA,$$

where B is the induction density in lines per sq. cm., and A is the cross-sectional area of the iron ring in sq. cms.

Hence $$B=\frac{rbT}{2Ak2\pi}\frac{10^8}{w_2}\Theta \text{ lines per sq. cm.}$$

To find the corresponding values of H the following relationship holds:

$$\frac{4\pi}{10}iw_1=Hl,$$

where l is the length in cms. of the mean path of the flux in the iron ring.

Hence $$H=\frac{4\pi iw_1}{10l}.$$

By taking a series of suitable values of the exciting current i, the corresponding series of values of B and H are determined. It is to be observed that when a new value of i is taken the current should be reversed several times, the galvanometer switch K being open, so that the flux in the iron may be carried through several cycles. This ensures that the flux has attained the correct value corresponding to the value of H for that current.

215. Determination of the Coefficient of Mutual Induction by means of the Ballistic Galvanometer.—In Fig. 274 let A be the primary and B the secondary of two mutually inductive coils. It is required to find the coefficient of mutual induction M.

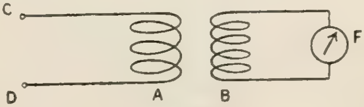


FIG. 274.

Let a current of i ampères flow in the coil A, and let the coil B be connected to a ballistic galvanometer F (if necessary, including a suitable resistance in the circuit of coil B). The flux linkages of coil B due to a current of i ampères in coil A is—

$$10^8 Mi \text{ c.g.s. lines (see § 206).}$$

If Q is the quantity discharged through the galvanometer circuit when the current i is broken, then—

$$Q = \frac{10^8 Mi}{r 10^8} = \frac{Mi}{r} \text{ coulombs (see § 205),}$$

where r is the total resistance in the circuit of coil B and the galvanometer.

Hence $$M = \frac{Qr}{i} \text{ henrys.}$$

216. The Fluxmeter.—It was explained in §§ 212, 213 that the measurement of magnetic field can be made by the ballistic galvanometer provided the flux linkages through the coil are varied so quickly that the change is complete before the moving system of the galvanometer has been appreciably displaced. The fluxmeter enables a magnetic field to be explored by means of a search coil in a similar way, but is not limited to the above restriction. The deflection of the fluxmeter is the same whether the flux change in the search coil be made quickly or reasonably slowly.

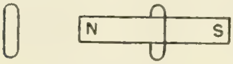


FIG. 275.

Thus, if the needle is originally at zero, and if a search coil connected to a fluxmeter is drawn over the N pole of a magnet (Fig. 275), a certain deflection will be obtained. When the coil is withdrawn from the S pole the same deflection in the opposite direction occurs.

The moving system of the fluxmeter is shown in Fig. 276. The moving coil is B, and the terminals of the coil are L and L¹. The suspension of the coil is a single cocoon silk fibre, the upper end of which is attached to a flat spiral spring to minimise shock. The current is led into and out from the coil by spirals of thin silver strip SS¹. The controlling force on the moving coil is extremely small, and this is an essential condition for the correct action of the fluxmeter. The coil and search coil, however, are of low resistance so that electro-magnetic damping of the coil is heavy.

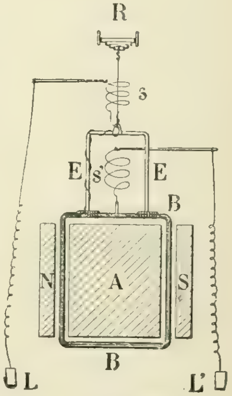


FIG. 276.

Theory.—

Let N be the flux in c.g.s. lines threading the search coil to which the fluxmeter is connected.

,, w be the number of turns in the search coil.

,, R be the resistance of the whole circuit of search coil and fluxmeter coil.

,, L be the self-induction of the whole circuit in henrys.

,, i be the current in ampères at any instant in the fluxmeter circuit.

,, $\frac{d\theta}{dt}$ be the angular velocity of fluxmeter coil at any instant.

The e. m. f. induced in the search coil at any instant will be (§ 197)—

$$e=\frac{w}{10^8}\frac{dN}{dt} \text{ volts.}$$

The back e. m. f. induced in the fluxmeter coil due to its movement with velocity $\frac{d\theta}{dt}$ in the field of the fluxmeter will be—

$$k\frac{d\theta}{dt} \text{ volts,}$$

where k is a constant of the instrument, and depends on the strength of the fluxmeter field and the number of turns in the fluxmeter moving coil.

Hence
$$\frac{w}{10^8}\frac{dN}{dt}-k\frac{d\theta}{dt}=iR+L\frac{di}{dt}.$$

Further, the electro-magnetic force action on the fluxmeter moving coil will be—

$$Fi,$$

where F is a constant of the fluxmeter.

The retarding force due to the air resistance on the fluxmeter moving coil will be—

$$A\frac{d\theta}{dt},$$

where A is some constant.

Hence
$$Fi-A\frac{d\theta}{dt}=K\frac{d^2\theta}{dt^2}$$

is the equation of motion of the fluxmeter coil when K is the moment of inertia of the coil.

Hence
$$K\frac{d^2\theta}{dt^2}=-A\frac{d\theta}{dt}+\frac{F}{R}\left[\frac{w}{10^8}\frac{dN}{dt}-k\frac{d\theta}{dt}-L\frac{di}{dt}\right].$$

Integrating with respect to the time and noting that the initial and final values of the velocity $\frac{d\theta}{dt}$ are both zero, and also the initial and final values of the current are zero, gives—

$$0=-\left(A+\frac{F}{R}k\right)\Theta+\frac{F}{R}\frac{w}{10^8}(N_2-N_1);$$

or the change in flux threading the search coil is—

$$N_2-N_1=\frac{\left(A+\frac{F}{R}k\right)\Theta}{\frac{F}{R}\frac{w}{10^8}}.$$

The constant damping A is relatively small and may be neglected, and usually N_1 or N_2 is zero—that is, the search coil is completely withdrawn from the field—hence the flux change is—

$$N=\frac{k10^8}{w}\Theta,$$

where Θ is the deflection of the fluxmeter needle.

The fluxmeter is calibrated to read flux change in a search coil of one turn per degree deflection of the fluxmeter coil, so that the flux change in a search coil of w turns is—

$$N=\frac{\kappa}{w}\Theta$$

the value of κ being supplied with the instrument.

By using search coils of different number of turns a wide range of values of magnetic fluxes may be measured.

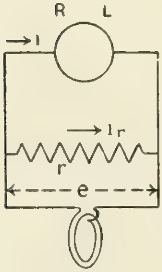


FIG. 277.

The range of the fluxmeter may also be varied by using a low-resistance shunt across the terminals of the fluxmeter, in which case the search coil may have a fixed number of turns.

For example, in Fig. 277 let the shunt of resistance r ohms be connected to the fluxmeter, and suppose the different symbols have the same significance as in the foregoing, then—

$$e=ii_r r.$$

Also
$$e-k\frac{d\theta}{dt}=iR+L\frac{di_1}{dt}$$

$$Fi-A\frac{d\theta}{dt}=K\frac{d^2\theta}{dt^2}.$$

Hence $$K\frac{d^2\theta}{dt^2} = -A\frac{d\theta}{dt} + \frac{F}{R}\left[e - k\frac{d\theta}{dt} - L\frac{di_1}{dt}\right].$$

Integrating with respect to t for the whole movement of the coil gives—

$$0 = -\left(A + k\frac{F}{R}\right)\Theta + \frac{F}{R}\int e\,dt.$$

Neglecting A as before, and noting that $\int e\,dt = \int i_r r\,dt = q_r r$, where q_r is the quantity of electricity in coulombs which passes through the shunt, then—

$$\Theta = \frac{q_r r}{k}.$$

If N is the flux change in the search coil of w turns to which this discharge is due, and if x is the resistance of the search coil

$$q_r = \frac{Nw}{(x+r)10^8} \text{ coulombs, approximately,}$$

assuming the resistance r of the shunt to be small compared with the resistance R of the fluxmeter coil.

Hence $$\Theta = \frac{Nwr}{k(x+r)10^8} \text{ coulombs,}$$

or $$N = \frac{10^8\,(x+r)k}{wr}\,\Theta \text{ c.g.s. lines,}$$

and by suitably choosing r, values of N over a wide range may be measured.

An important application of the fluxmeter is the measurement of the flux in the magnetic circuit of dynamo-electric machines.

For example, suppose Fig. 278 represents part of the field system and armature of a dynamo. By winding two coils A and A_1 round the yoke and connecting the coils in series, the flux in the yoke when the field coils are excited may be measured by the fluxmeter. Further, if a coil B be laid on the armature so as to embrace the flux entering the armature from one pole, this flux may also be measured by the fluxmeter, and the ratio—

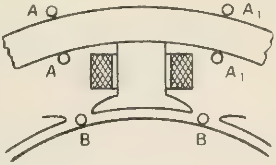


FIG. 278.

$$\frac{\text{Total flux through coils A and A}_1}{\text{Flux through coil B}} = \sigma,$$

where σ is the magnetic leakage coefficient.

Owing to the large self-induction of the field coils of a dynamo, the rise of flux when the coils are excited is relatively slow, and for this reason a ballistic galvanometer would not be suitable for measuring this flux.

217. The Duddell Oscillograph.—This instrument is a specialised form of moving coil galvanometer in which a very light coil is tightly stretched in a powerful magnetic field, the free undamped period of vibration of the moving system being extremely small—viz., $\frac{1}{10000}$ second in the high-frequency type of oscillograph. The moving system is damped by being immersed in oil, and will follow accurately alternating wave forms up to a frequency of 300 cycles per second.

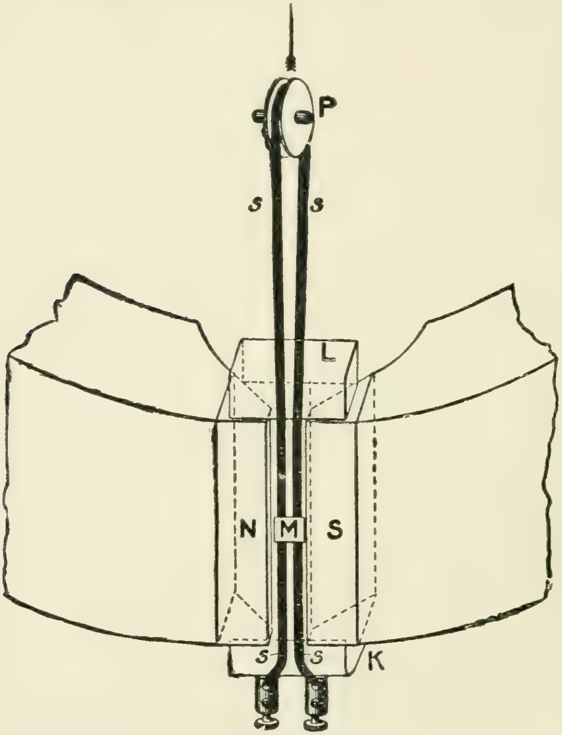


Fig. 279.

Each instrument has two independent moving systems, and can thus be used to trace two wave forms simultaneously—*e.g.*, potential difference and current waves.

The arrangement is shown diagrammatically in Fig. 279, which shows a single loop of strip conductor stretched in the air gap between the poles of a powerful magnet. The strip passes over an ivory pulley P, and a spiral spring attached to the pulley serves to keep a uniform tension on the strips. A guide piece L limits the length of the vibrating portion of the strip to the part actually in the magnetic field.

A small mirror M is secured to the two sides of the loop, and when a current is passed round the loop the two sides move in opposite directions across the field and so cause the mirror M to turn. The instrument is provided with a photographic equipment, in which the film is moved at a constant speed in a direction at right angles to the movement of a spot of light reflected from the mirror M. In this wave a curve showing the time variation of the current in the strip may be recorded.

With the most sensitive type of oscillograph currents of about 0·001 ampère may be detected. The self-induction of the moving system is negligibly small.

In Chapter XXI., Fig. 429, are shown some photographic records (or "oscillograms") taken by means of this instrument.

218. Electro-Dynamometer Instruments—I. *Wattmeter*—All electro-dynamometer instruments consist essentially of two concentric coils—viz., a fixed coil and a movable coil. In a wattmeter the fixed or current coil is connected in series with the current in the circuit, and the movable or pressure coil is connected across the mains, a suitable resistance being connected in series if the supply p. d. is too high to be safely connected directly to the pressure coil. The arrangement of the connections is shown diagrammatically in Fig. 280.

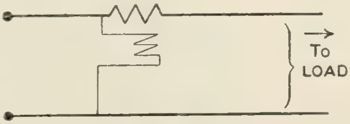


Fig. 280.

Let I be the current in the circuit and V the supply p. d. Then if r is the resistance of the pressure coil circuit the current in this coil will be $i=\frac{V}{r}$. For any given relative position of the two coils of the instrument the turning moment on the moving coil will be—

$$KiI=K\frac{V}{r}I,$$

where K is a constant.

Hence the turning moment on the moving coil is proportional to the power supplied to the circuit.

In the early Siemen's type of instruments, the relative position of the two coils was maintained constant, and the turning moment on the pressure coil balanced by twisting a torsion head from which the moving coil was suspended, the two ends of a spiral spring being secured to the torsion head and moving coil respectively. In this case the torsion on the spring was read on a graduated scale by means of a pointer fixed to the torsion head. For equilibrium the twist of the head would then be directly proportional to the product $i\mathrm{I}$, so that the scale was uniformly divided throughout its entire range.

For practical purposes, however, it is very inconvenient to adjust a torsion head each time a reading is taken, and hence the instruments are now made with a pointer fixed direct to the moving coil, and the deflection of the pointer as read on a graduated scale gives the watts supplied to the consumer.

In Fig. 281 is given an illustration of a Weston wattmeter, the resistances connected in series with the pressure coil being shown arranged under the scale of the instrument.

Since the pressure coil moves with the pointer the relative positions of the two coils vary with the reading of the instrument, and consequently the graduation of the scales are not uniform, and the instrument must be calibrated throughout its range by comparison with some standard instrument.

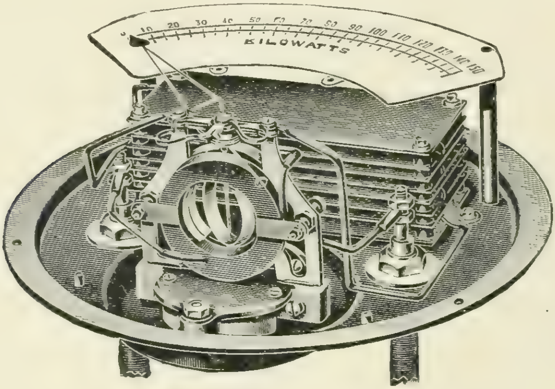


Fig. 281.

A dynamometer wattmeter can be used on direct-current or alternating-current circuits. In the latter case the wattmeter indicates the mean power supplied to the circuit. The current in the pressure coil, when used on alternating-current circuits is—

$$i = \frac{\mathrm{V}}{\sqrt{r^2 + (\mathrm{L}\omega)^2}} \text{ (see Chapter XVII.)},$$

where r is the resistance and L the inductance of the pressure coil circuit, and $\omega = 2\pi$ frequency.

Hence, in order that the readings of the wattmeter should be independent of the frequency of the supply, the quantity $\mathrm{L}\omega$ must be made vanishingly small as compared with r—that is to say, the inductance of the wattmeter pressure coil must be made as small as possible.

For more detailed information as to the various methods of wattmeter measurements, reference should be made to Chapter XVIII., Section V.

II. *Voltmeters.*—In these instruments the two coils are connected in series so that the same current flows round both coils. If R is the total resistance of the two coils in series, the current in the coils will be—

$$i = \frac{V}{R}.$$

The turning moment on the moving coil will be proportional to i^2—that is, to $\left(\frac{V}{R}\right)^2$, and hence the scale of the instrument may be graduated to read volts directly. These instruments are made so that the value of the self-induction of the coil is as small as possible, the current in the coils being then directly proportional to the supply pressure and practically independent of the frequency of supply (see previous paragraph on Wattmeters).

III. *Ammeters.* — Electro-dynamometer instruments constructed for measuring current are arranged with the fixed and moving coils in series, as in the voltmeter and the instrument connected in series with the mains. The turning moment on the movable coil is again proportional to the square of the current in the coils, and the scale is graduated to indicate the current in ampères. Usually the fixed coil of the ammeter is made in two sections which, by means of external connectors, may be connected in series or in parallel. When the two sections of the fixed coil are connected in parallel the range of the instrument is double the range when the two sections are connected in series.

Usually a shunt is fitted inside the instrument so that only a fraction of the total current passes through the coils.

IV. *Frequency Meters.*—The precise action of this type of electro-dynamometer instrument will be better understood after reading Section V. The object of the instrument is to indicate the frequency of supply of alternating current mains. In the Weston type of frequency meter there are two fixed coils, as shown by 1 and 2 in Fig. 282. These coils are connected in series with each other, and one end of the series arrangement is connected through an inductance X_1 to one main, and the other end through a resistance R_2 and inductance X to the other main. A resistance R_1 and inductance X_2 are in series, and connected across the common junction of R_2 and X and to the common junction of X_1 and the main. The common junction of R_1 and X_2 is connected to the common junction of the coils 1 and 2 by means of the lead A.

A movable coil (shown by a heavy black line in Fig. 282) is pivoted centrally with the fixed coils 1 and 2, but has no electrical connections to the rest of the circuit, and has no spring control.

The arrangement of fixed coils 1 and 2, inductances X_1 and X_2, and resistances R_1 and R_2, shown in Fig. 282, from a Wheatstone bridge circuit (compare Fig. 178), the lead A corresponding to the galvanometer bridge connection. At the normal frequency of the circuit the bridge is balanced and the moving coil occupies the position shown in Fig. 282. A change in frequency produces a change in the space position of the resultant field of the coils 1 and 2, and the movable coil shifts correspondingly. Every change in frequency affects the current in *each* of the coils 1 and 2, increasing the field strength of one coil and decreasing that of the other.

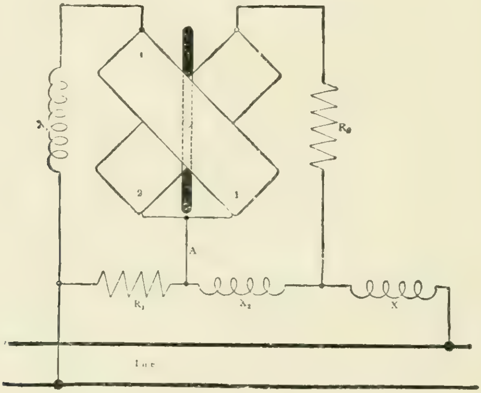


FIG. 282.

219. Vibrating Reed Frequency Meter.—This type of frequency meter depends for its action on the mechanical resonance between a reed fixed at one end and subjected to an alternating force produced by a magnetic field excited from the main supply. Actually the arrangement consists of a number of thin steel reeds mounted on a bar. This bar is flexibly supported from the base of the instrument, and secured to the bar is a soft iron armature which is subjected to the magnetic force of an alternating current electro-magnet excited by current from the main supply. If the frequency of the alternating magnetic pull is the same as the natural frequency of any of the reeds secured to the bar, that reed will be set oscillating with considerable amplitude, whilst the other reeds will be practically unaffected.

In Figs. 283 and 284 are shown respectively the complete instrument and the internal arrangement of a reed frequency meter as manufactured by Messrs. Elliott Bros., Ltd. To the

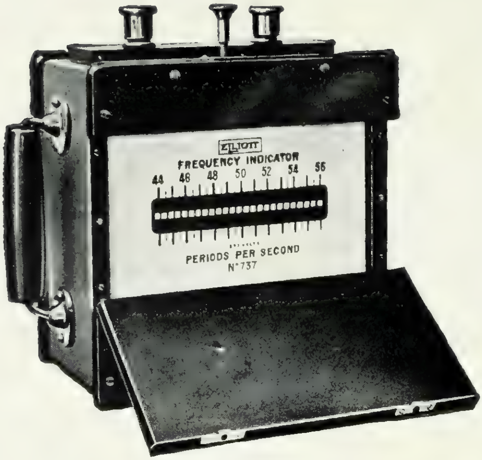


Fig. 283.

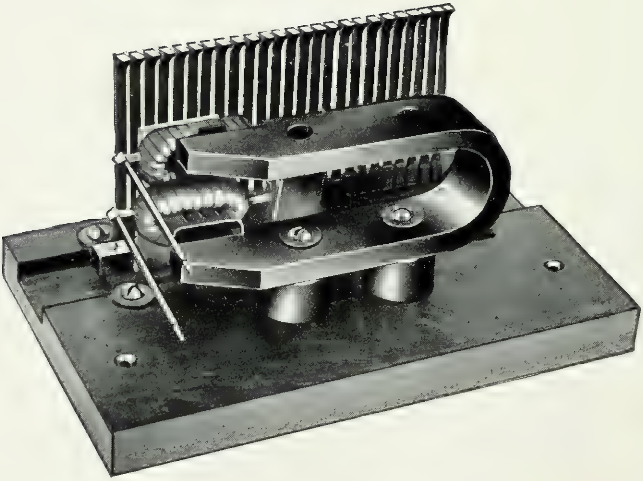


Fig. 284.

To face p. 317.

end of each reed is fixed a white enamelled blob, so that the vibration of the reed is easily seen.

In the vibrating reed frequency meters, illustrated in Figs. 283 and 284, the coils are wound on the poles of a permanent magnet; the induction density due to the permanent magnet is about equal to the maximum value of the flux density due to the alternating current supplied to the coils. The resultant flux density acting on the armature secured to the reed bar is thus—

$$B_M + \bar{B} \sin wt = \bar{B}(k + \sin wt),$$

where B_M is the induction density due to the permanent magnet and $k = \frac{B_M}{\bar{B}}$.

As was shown in § 117*a*, Chapter VIII., the magnetic pull on the armature will be proportional to the square of the flux density in the air gap—that is,

$$= K\bar{B}^2 (k + \sin wt)^2 = K\bar{B}^2 (k^2 + 2k \sin wt + \sin {}^2wt),$$

where K is some constant.

That is to say, the expression for the magnetic pull contains a term of the same frequency as that of the alternating current supply. In this case, therefore, if the natural frequency of any of the reeds has *the same* value as that of the alternating current supply, that reed will vibrate with a large amplitude.

In Fig. 284 the reed bar is secured at each end to a strip of spring steel, which thus forms flexible supports for the bar. The horseshoe permanent magnet is provided with a coil wound on each pole, which coils are arranged in series and connected to the alternating current supply. The flat strip armature faces the poles of the magnet, and is secured to the reed bar so that the alternating magnetic pull on the armature is transmitted to the bar supporting the reeds.

The strip of steel shown near the middle of the horse-shoe magnet is secured at one end to the reed bar. When the instrument is mounted in its case a fine pitched screw butts against the top of this strip, and allows a fine adjustment of the armature relatively to the poles of the magnet. The object of this adjusting device is to compensate for variations in the supply voltage, so that if a variation up to 20 per cent. on either side of the mean value of the supply voltage occurs the instrument can be adjusted to suit the voltage.

By mounting two armatures on the reed bar and actuating one by the resultant of the superposition of a steady flux and the flux due to an alternating current—that is to say, by a "polarised" magnet—and actuating the other armature by the flux due to an electro-magnet only, the range of frequencies of the instrument may be doubled for a given number of vibrating reeds.

Vibrating reed instruments can be made to measure frequencies up to 700 cycles per second.

In some reed instruments the permanent magnet is dispensed with, and in such cases the following considerations apply:

If the exciting current is of sine wave form, so that—

$$i = \bar{I} \sin \omega t,$$

then the value of the induction density in the air gap at any moment will be—

$$B = \bar{B} \sin \omega t.$$

The pull on the reed bar will therefore be proportional to—

$$\bar{B}^2 \sin^2 \omega t;$$

or

$$\text{pull} = K\bar{B}^2 \sin^2 \omega t = K\bar{B}^2 \left(\frac{1 - \cos 2\omega t}{2}\right)$$

$$= k\bar{B}^2 - k \cos 2\omega t.$$

That is to say, the magnetic pull will be a steady pull $k\bar{B}^2$, and superposed on this steady pull an alternating pull of *twice* the frequency of the exciting current. It is therefore necessary to tune the reeds to twice the frequency of the alternating current supply.

The scale of the instrument is marked, however, to read the frequency directly in periods per second.

220. Moving Coil Direct Current Instruments.—These instruments are constructed on precisely the same principle as the moving coil galvanometer shown in Fig. 268, the moving coil being now fitted with a pointer which moves over a graduated scale.

Voltmeters are provided with a set of series resistances which enable the full scale of the instrument to be used for a wide range of voltages, and ammeters are provided with a set of shunts so that the full scale of the instrument may be used for a wide range of currents.

221. Hot-Wire Instruments.—The method of measuring potential difference by the increase of length of a wire heated by the passage of the current when connected to the mains of which the p. d. is to be measured was devised by Cardew. The form of the instrument designed by Cardew is now obsolete, but the principle is still used, and such instruments have a very wide application. Messrs. Johnson and Phillips are the makers in this country of the modern type which was developed by Messrs. Hartmann and Braun.

222. Moving Iron Alternating-Current Instruments.—This type of instrument depends for its action on the tendency of a

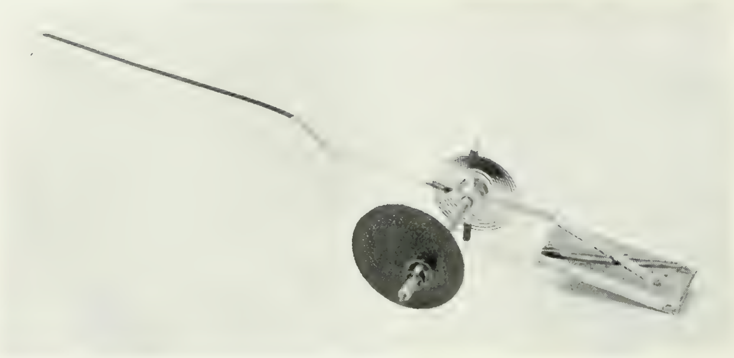


FIG. 285.

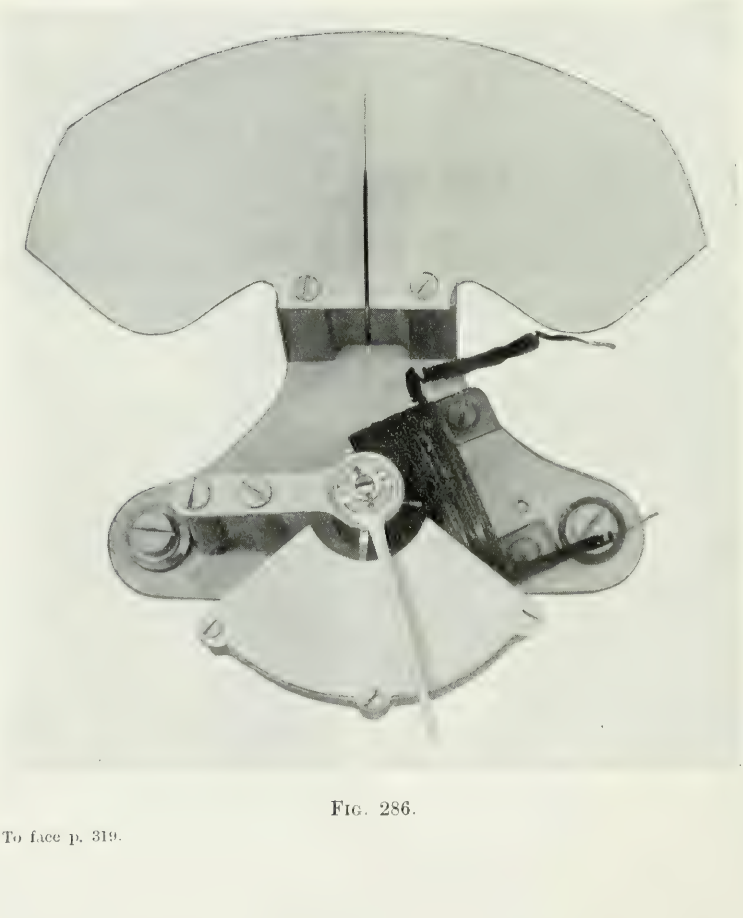


FIG. 286.

To face p. 319.

piece of iron to move from a weak part of a magnetic field to a strong part.

In Figs. 285 and 286 is shown such an instrument as made by Messrs. Elliott Brothers, Ltd.

The moving parts of the instrument are shown separately in Fig. 285, and consist of the pointer, the air-damping vane, and a thin disc of alloy steel mounted eccentrically on a spindle.

In Fig. 286 are shown, mounted in position, the moving parts and the solenoid in which the alternating current to be measured flows.

When the solenoid is excited the flat alloy steel disc is acted on by a magnetic force and turns about its eccentrically mounted axis.

The alloy steel for the disc is specially selected so that the hysteresis effect is small, and the instrument is very accurate, of light weight, and practically independent of the frequency of supply over a wide range of frequencies.

SECTION V

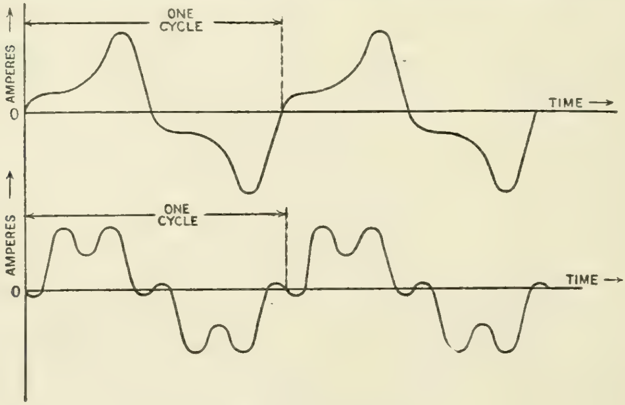
ALTERNATING CURRENTS

CHAPTER XVII

DEFINITIONS—ALTERNATING CURRENTS OF SINE WAVE FORM.

223. Alternating Current.—An alternating electric current is one which passes through a complete cycle of changes periodically. In the generally accepted definition of an alternating current each cycle consists of two half-cycles, during one of which the current is usually chiefly or entirely positive, and during the other chiefly or entirely negative, the shape of the negative half-cycle being an exact reproduction of the shape of the positive half-cycle.

A complete cycle of changes is called a *wave* of current.



Figs. 287, 288.

In Figs. 287 and 288 are shown waves of two such alternating currents, the time being plotted as abscissæ, and the zero of time chosen in each case as the moment when the current passes through one of its zero values.

A somewhat different type of alternating current wave is shown in Fig. 289, in which one half-cycle consists of positive and negative values, the other half-cycle consisting of an exactly similar series of positive and negative values (see also § 261, Fig. 391, Chapter XIX.).

If the current vary from moment to moment, but does not change sign during the cycle (*i.e.*, is wholly positive and negative), the current is termed a *pulsating* current. In Fig. 290 is given an example of such a current.

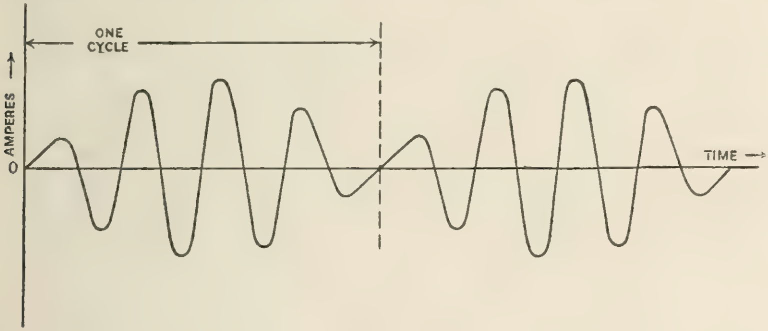


Fig. 289.

Referring to Figs. 287-289, the time T of a complete cycle is called the *period* or *periodic* time of the current.

The number of complete cycles through which the current passes in one second is called the **frequency** or **periodicity** of the alternating current, and is usually denoted by f.

Hence $f=\frac{1}{T}$ cycles per second,

and $T=\frac{1}{f}$ second.

The usual value of the frequency of alternating currents for lighting and power distribution is of the order of from 25 to 60 cycles per second.

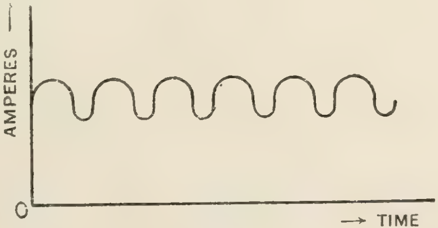


Fig. 290.

For special purpose frequencies up to 500 cycles per second are becoming usual, and alternating current generators for such frequencies are readily obtainable.

For wireless telegraphy the frequencies used are commonly of the order of several hundred thousands per second.

224. Alternating Currents of Sine Wave Form.—Suppose the current varies with the time according to a sine law—viz.:

$$i=\bar{I} \sin \omega t \text{ ampères},$$

where t is the time in seconds, ω is a constant, and $\bar{I}$ the maximum value, the current attains during a cycle. Such a wave form is taken as the standard in electrical engineering. Of all possible wave forms it is the simplest to deal with mathematically, and it can be proved that any continuous periodic function can be analysed into a series of sine waves of different frequencies (see Chapter XIX.).

The e. m. f. wave forms of modern alternators are very approximately sine curves, and, as will be seen later, the sine wave form of e. m. f. has the advantage that there is no component of high frequency which might give rise to accidental resonance effects in circuits containing inductance and capacity (see § 230).

Suppose a circle be drawn (Fig. 291) of radius O_1A representing to scale the maximum value $\bar{I}$ of the current, and suppose O_1A revolves about O with a uniform angular velocity ω radians per second in a counter-clockwise direction.

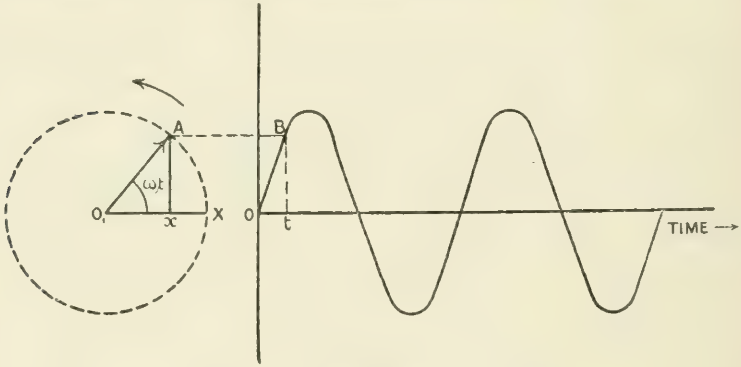


FIG. 291.

Let O_1X be the starting position of O_1A (*i.e.*, let the zero of time be reckoned from the moment when O_1A coincides with O_1X). At the time t, the angle which O_1A makes with O_1X will therefore be—

$$\omega t \text{ radians.}$$

From A draw Ax perpendicular to O_1X.

Then
$$Ax = O_1A \sin \omega t$$
$$= \bar{I} \sin \omega t;$$

or Ax represents the value at the time t of the alternating current of sine wave form, of which the maximum value is $\bar{I}$.

Hence, if the values of t be plotted as abscissæ, the corresponding projections of O_1A on the vertical will give the sine wave form of current.

Since the angular velocity of O_1A is ω radians per second it follows that O_1A makes $\frac{\omega}{2\pi}$ revolutions per second—*i.e.*, the current passes through $\frac{\omega}{2\pi}$ cycles in one second.

Hence the relationship between the frequency f and the angular velocity ω is—

$$f=\frac{\omega}{2\pi}.$$

The time of one complete cycle is—

$$\mathrm{T}=\frac{1}{f}=\frac{2\pi}{\omega} \text{ second.}$$

In the following the symbols $\bar{\mathrm{I}}$, $\bar{\mathrm{V}}$, $\bar{\mathrm{E}}$ will be used to represent the maximum values (or amplitudes) of sine waves of current, potential difference, and e. m. f. respectively.

225. Vector Method of representing Sine Wave Forms of Alternating Current.—In Fig. 292 let the length of the vector OA represent the maximum value $\bar{\mathrm{I}}$ of a sine wave form of current, the direction of the vector being from O to A. If the vector rotate with uniform angular velocity ω in a counter-clockwise direction it has been shown in the preceding paragraph that the projection of the vector on the vertical represents the instantaneous value of the current $i=\bar{\mathrm{I}}\sin\omega t$ at the time t, the zero of time being taken as the moment at which OA coincides with OX.

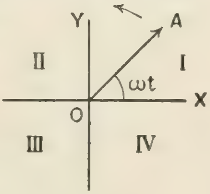


FIG. 292.

When the vector OA is in quadrant I or quadrant II the current is positive, and when OA is in quadrant III or quadrant IV the current is negative. From the numerous examples given in subsequent pages it will be seen that the vector method forms a very powerful and simple means of dealing with alternating current problems.

The remarks made in this and preceding paragraphs with reference to alternating currents apply also to alternating e. m. f.'s and alternating p. d.'s of sine wave forms.

226. P. D. necessary to send an Alternating Current of Sine Wave Form through a Non-inductive Resistance.—Let R ohms be the value of the resistance and let the current be $i=\bar{\mathrm{I}}\sin\omega t$ ampères (Fig. 293).

The instantaneous value of the p. d. across the resistance will then be (§ 140)—

$$v=i\mathrm{R}=\bar{\mathrm{I}}\mathrm{R}\sin\omega t \text{ volts;}$$

or $$v=\bar{\mathrm{V}}\sin\omega t \text{ volts,}$$

where $$\bar{\mathrm{V}}=\bar{\mathrm{I}}\mathrm{R} \text{ volts.}$$

The p. d. across the resistance is of sine wave form, and is proportional at every instant to the current at that instant.

Example.—Let

$$R=5 \text{ ohms}: f=50: \omega=2\pi\times 50=314$$

$$T=\frac{1}{50}=0{\cdot}02 \text{ second}$$

$$i=20\sin\omega t=20\sin 314t \text{ ampères.}$$

Then
$$v=20\times 5\sin 314t$$
$$=100\sin 314t \text{ volts.}$$

The waves of current and p. d. are shown in Fig. 294. The two waves pass through their zero values simultaneously, and also reach their maximum values simultaneously. The two waves are therefore said to be *in phase,* and it is a characteristic feature of a non-inductive resistance that the p. d. across the resistance is in phase with the current flowing in the resistance.

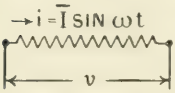


Fig. 293.

The vector diagram for the current and p. d. in this case is given in Fig. 295. The vector OA represents the current and the vector OB the p. d., the two vectors coinciding in direction.

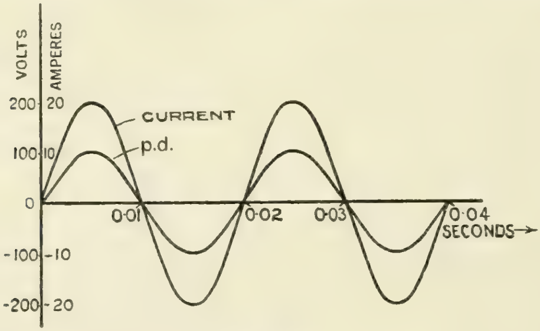


Fig. 294.

227. P. D. necessary to send an Alternating Current of Sine Wave Form through an Inductance Coil of Negligibly Small Resistance.

—Let the inductance of the coil be L henrys (Fig. 296). It was shown on p. 284, Section IV., that if a current of i ampères flowing in a coil of inductance L henrys varies at the rate $\frac{di}{dt}$, a back e. m. f. of $-L\frac{di}{dt}$ volts is developed which tends to oppose the variation of the current.

If the current is

$$i=\bar{I}\sin\omega t \text{ ampères,}$$

the back e. m. f. developed in the coil is

$$-L\frac{di}{dt} = -L\bar{I}\omega \cos \omega t \text{ volts},$$

the direction of this back e. m. f. at any instant being such as to oppose the change of current at that instant. Hence in order to force the current to vary as defined by the expression $\bar{I} \sin \omega t$, a p. d. must be applied to the terminals of the coil, such as to be exactly *equal and opposite* to the back e. m. f $-LI\omega \cos \omega t$. The applied p. d. is therefore—

$$v = \bar{I}L\omega \cos \omega t \text{ volts}$$

$$= \bar{V} \cos \omega t \text{ volts};$$

where $\bar{V} = \bar{I}L\omega$ volts.

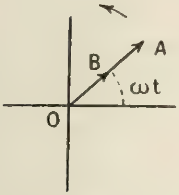


FIG. 295.

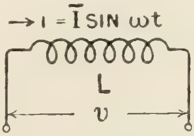


FIG. 296.

The quantity $L\omega$ is termed the **inductive reactance** of the coil, and is measured in ohms.

Example.—Let

$$L = 0{\cdot}01 \text{ henry}: f = 50: \omega = 2\pi f = 314$$

$$L\omega = 3{\cdot}14 \text{ ohms}: i = 30 \sin \omega t = 30 \sin 314t \text{ ampères}.$$

Then $v = \bar{V} \cos \omega t = \bar{I}L\omega \cos wt$ volts,

or $v = 94 \cos 314t$ volts.

In Fig. 297 the waves of current and applied p. d. are shown, and it will be seen that the current reaches its maximum positive value one-quarter of a period (*i.e.*, $\frac{\pi}{2}$ radians or 90° in the vector diagram) later than the applied p. d. The current is therefore said to *lag* on the applied p. d by one-quarter of a period or 90°.

It is characteristic of a purely inductive coil (*i.e.*, a coil of negligibly small resistance) that the sine wave of alternating current *lags* by 90° on the applied sine wave of p. d.

The vector diagram of this case is shown in Fig. 298, in which OA is the vector of current and OB the applied p. d.

It follows at once that if the applied p. d. is

$$v=\overline{V}\sin\omega t,$$

the current will be—

$$i=\frac{\overline{V}}{L\omega}\sin(\omega t-90^\circ)=-\frac{\overline{V}}{L\omega}\cos\omega t;$$

that is, lagging by a quarter period on the applied p. d.

It is of interest to consider briefly the physical meaning of the fact that the alternating current in an inductive coil lags by 90° on the applied p. d. Take the instant at which the current is a positive maximum—*e.g.*, time A (Fig. 297). At this moment the current, having reached a maximum value, is momentarily steady, and since the resistance of the coil is zero no p. d. is necessary to keep the current flowing, and the wave of applied p. d. therefore passes through its zero value at the time A. The current then begins to diminish and consequently the magnetic flux threading the coil diminishes.

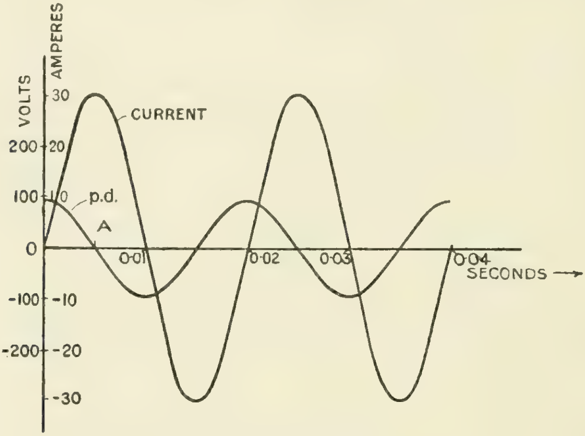


FIG. 297.

According to Lenz's Law (§ 195), however, a back e. m. f. is induced by the shrinking flux, and tends to keep the current flowing undiminished in strength. Hence the applied p. d., which must oppose this back e. m. f., is then negative. By similar reasoning the direction of the applied p. d. at every instant may be deduced.

228. The Alternating Current through a Condenser due to a Sine Wave Form of Applied P. D.—Let C be the capacity of the condenser (Fig. 299) in farads (§ 36), and let the applied p. d. be given by

$$v=-\overline{V}\cos\omega t \text{ volts (see Fig. 300).}$$

If at any time t the quantity (in coulombs) of the charge of the condenser is Q, the p. d. across the condenser plates is—

$$v=\frac{Q}{C} \text{ volts.}$$

The current through the condenser is (§ 132)—

$$i=\frac{dQ}{dt} \text{ ampères,}$$

$$=C\frac{dv}{dt} \text{ ampères,}$$

the condenser being assumed to be uncharged when $t=o$.

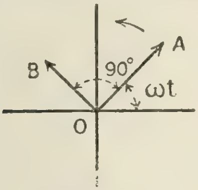


FIG. 298.

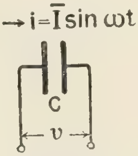


FIG. 299.

That is $i=\omega C\bar{V}\sin\omega t$ ampères,

or $i=\bar{I}\sin\omega t$ ampères,

where $\bar{I}=\omega C\bar{V}$,

or $\bar{V}=\dfrac{\bar{I}}{\omega C}$.

The quantity $\dfrac{1}{\omega C}$ is termed the **capacity reactance**, and is measured in ohms.

Example.—Let $C=200\times10^{-6}$ farads $=200$ micro-farads: $f=50$: $w=2\pi f=314$: $\dfrac{1}{\omega C}=16$ ohms: $v=-\bar{V}\cos\omega t=-128\cos 314\,t$.

Then $i=\omega CV\sin\omega t$ ampères,

$$=\frac{314\times200\times128}{10^6}\sin\omega t \text{ ampères,}$$

that is $i=8\sin 314\,t$ ampères.

The waves of current and applied p. d. for this case are shown in Fig. 300. It will be observed that the current reaches its maximum value one quarter period before the applied p. d. —*i.e.*, the current *leads* on the applied p. d. by 90°. It is a characteristic feature of a condenser in an alternating current circuit that the sine wave of current leads by 90° on the applied p. d., and this is a valuable feature which has important practical applications.

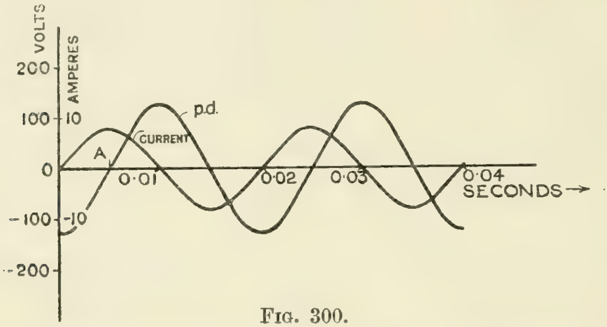


Fig. 300.

The vector diagram for the case of an alternating current in a condenser is given in Fig. 301. The current vector is OA and the vector of applied p. d. is OB.

Since $\bar{V}=\frac{\bar{I}}{\omega C}$ or $\bar{I}=\bar{V}\omega C$ it follows that if ωC is large a small value of the applied p. d. will produce a large current in the condenser. The higher the frequency the larger the current which a given applied p. d. will produce through a given condenser. Important practical consequences of this result arise when the wave of applied p. d. comprises harmonics (see Chapter XIX.).

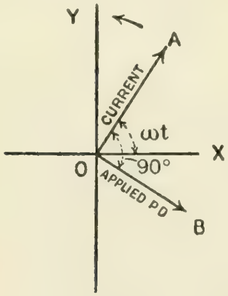


Fig. 301.

If the applied p. d. at the condenser terminals is given by the expression—

$$v=\bar{V}\sin wt \text{ volts,}$$

it will be clear from the foregoing that the current will be given by the expression—

$$i=\bar{I}\sin(wt+90°)$$
$$=\bar{V}wC\sin(wt+90°)$$
$$=V\bar{w}C\cos wt \text{ ampères.}$$

The physical meaning of the fact that the current through the condenser leads by 90° on the applied p. d. may be seen from the following considerations:

In Fig. 302 let the positive direction of the applied p. d. be such as to charge the plate A positively and the plate B negatively. The positive direction of the current will then be from A to B through the condenser.

Consider the moment immediately after time 0·01 second (Fig. 300). The current in the circuit begins to flow in the negative direction—*i.e.*, from plate A through the generator G to the plate B. This means that at the moment 0·01 second the condenser has its maximum positive charge, and the next moment the charge has diminished. Since the applied p. d. is directly proportional to the charge, the applied p. d. at time 0·01 second (Fig. 300) has its maximum positive value. Similarly at the time 0·02 second (Fig. 300)the current is zero and increasing in a positive direction, which means that the condenser charge at this moment has its maximum negative value, and the current begins to flow from B through the generator G to the plate A (Fig. 303)—*i.e.*, in the positive direction.

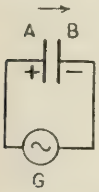


FIG. 302.

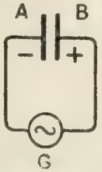


FIG. 303.

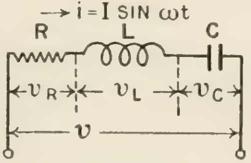


FIG. 304.

229. Applied P. D. necessary to send an Alternating Current of Sine Wave Form through a Resistance, Inductance, and Condenser in Series.—In Fig. 304 let v_R be the p. d. across the resistance R, v_L the p. d. across the inductance L, v_C the p. d. across the condenser C, and let $i=\bar{I}\sin\omega t$ ampères, the current through the series arrangement at any moment t.

Then
$$v_R=\bar{I}R\sin\omega t\text{ volts,}$$
$$v_L=\bar{I}L\omega\cos\omega t\text{ volts,}$$
$$v_C=-\frac{\bar{I}}{\omega C}\cos\omega t\text{ volts.}$$

The applied p. d. across the series arrangement at the moment considered is—
$$v=v_R+v_L+v_C=\bar{I}\left[R\sin\omega t+\left(L\omega-\frac{1}{\omega C}\right)\cos\omega t\right]\text{ volts;}$$

that is
$$v=\bar{I}Z\left\{\frac{R}{Z}\sin\omega t+\frac{L\omega-\dfrac{1}{\omega C}}{Z}\cos\omega t\right\}\text{volts,}$$

where
$$Z=\sqrt{R^2+\left(L\omega-\frac{1}{\omega C}\right)^2}.$$

That is— $v = \bar{V} \sin(\omega t + \phi)$ volts,

where $\bar{V} = \bar{I}\sqrt{R^2 + \left(L\omega - \frac{1}{\omega C}\right)^2}$

and $\tan \phi = \dfrac{L\omega - \frac{1}{\omega C}}{R}$.

The quantity $\sqrt{R^2 + \left(L\omega - \frac{1}{\omega C}\right)^2}$, denoted by the letter Z, is termed the **impedance** of the circuit. The quantity $\left(L\omega - \frac{1}{\omega C}\right)$ is termed the **reactance** of the circuit, and is denoted by the letter X.

The above is the most general case of an alternating current series circuit, and each of the cases considered in §§ 226, 227, and 228 may be deduced from this general result.

It is to be observed that if the condenser is out of the circuit (or, what is the same thing, is short-circuited by a piece of thick copper wire), the condition is expressed by putting C equal to infinity in the formula. Thus, for a circuit containing an inductance L and resistance R in series—

$$C = \text{infinity and } \tan \phi = \frac{L\omega}{R};$$

$$v = \bar{I}\sqrt{R^2 + (L\omega)^2} \sin(\omega t + \phi) \text{ volts.}$$

The results may be expressed generally as follows:

If a sine wave of alternating current flows in a series connected circuit, the applied p. d. must be of sine wave form and displaced in phase relatively to the current wave by an angle ϕ where

$$\tan \phi = \frac{L\omega - \frac{1}{\omega C}}{R}.$$

If $L\omega$ is $> \frac{1}{\omega C}$ the current *lags* on the applied p. d., and ϕ in the above formula is positive.

If $L\omega$ is $< \frac{1}{\omega C}$ the current *leads* on the applied p. d., and ϕ in the above formula is negative.

If $L\omega = \frac{1}{\omega C}$ the current is *in phase* with the applied p. d. This is a special and important case which will be considered more fully in § 230.

Since for the current $i=\bar{I} \sin \omega t$ ampères, the necessary applied p. d. is $v=\bar{V} \sin (\omega t+\phi)$ volts, it follows at once that if the applied p. d. is $v=\bar{V} \sin \omega t$ volts the current will be $i=\bar{I} (\sin \omega t-\phi)$ ampères.

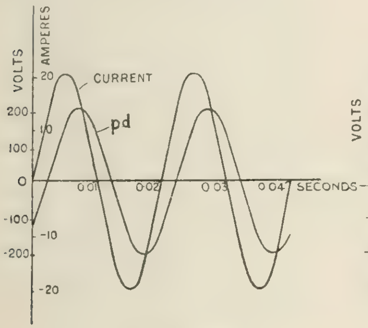


FIG. 305.

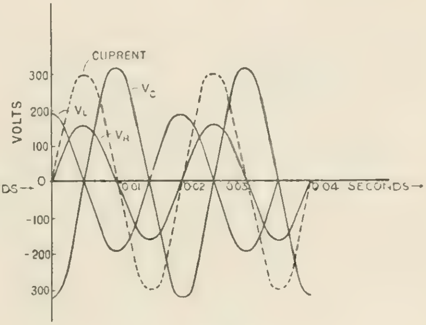


FIG. 306.

A more general treatment of the case of an alternating current series circuit is given in Chapter XXI. on "Transient Currents."

Example.—Let L=0·03 henry: C=200 micro-farads: $f=50$: $w=2\pi f=314$: R=8 ohms: $L\omega=9{\cdot}42$ ohms: $\frac{1}{\omega C}=16$ ohms: $L\omega-\frac{1}{\omega C}=-6{\cdot}58$ ohms: $\sqrt{R^2+\left(L\omega-\frac{1}{\omega C}\right)^2}=10{\cdot}3$: $\tan\phi=\frac{-6{\cdot}58}{8}=-0{\cdot}825$; $\phi=-39°\ 30'$. Hence the current *leads* on the applied p. d. by 39° 30′.

If $i=20 \sin \omega t=20 \sin 314t$ ampères
$v=206 \sin (314t-39°\ 30')$ volts.

In Fig. 305 the waves of current and applied p. d. for this case are shown. The same result is, of course, obtained if the component p. d. waves v_R, v_L, and v_C are drawn and the corresponding ordinates added.

Thus $v_R=160 \sin 314t$:
$v_L=188{\cdot}4 \cos 314t$:
$v_C=-320 \cos wt$.

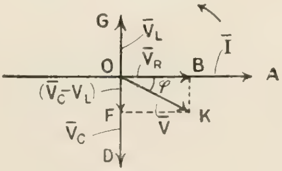


FIG. 307.

These component waves of p. d. are shown in Fig. 306.

The vector diagram for such a series circuit is shown in Fig. 307.

The current vector is OA, the vector of p. d. $\bar{V}_R$ is OB, the vector of p. d. $\bar{V}_L$ is OG, and the vector of p. d. $\bar{V}_C$ is OD. OK is the vector sum of OB, OG, and OD, and is the vector of applied p. d. $\bar{V}$. The current vector OA leads by the angle ϕ on the vector OK of applied p. d.

230. Resonance.—As was pointed out in § 229, a special and important case of a series circuit is one in which $L\omega = \frac{1}{\omega C}$. In this case the angle ϕ is zero—that is, the current is in phase with the applied p. d.

If $$L\omega = \frac{1}{\omega C},$$

then $$\omega^2 = \frac{1}{LC};$$

or $$\omega = \frac{1}{\sqrt{LC}}$$

and $$f = \frac{\omega}{2\pi} = \frac{1}{2\pi\sqrt{LC}}.$$

As will be seen later (Chapter XXI.), $\frac{1}{2\pi\sqrt{LC}}$ is the frequency at which a current, started in a closed circuit of small resistance of inductance L henrys and capacity C farads, would continue to oscillate; or, in other words, $\frac{1}{2\pi\sqrt{LC}}$ is the *natural frequency* of the circuit. If the applied p. d. has the frequency $f = \frac{1}{2\pi\sqrt{LC}}$ the circuit is said to be in *resonance* with the supply frequency. It will also be seen later (p. 444) that for relatively moderate values of the resistance R in the circuit the natural frequency is practically independent of the resistance of the circuit.

The condition of resonance is—

$$L\omega = \frac{1}{\omega C},$$

since $$v_L = L\omega \bar{I} \cos wt$$

$$v_C = \frac{\bar{I}}{-\omega C} \cos \omega t,$$

it follows that $$v_L = -v_C$$

and $$v = v_R + v_L + v_C = v_R;$$

that is $$v = \bar{I}R \sin \omega t \text{ volts}.$$

Hence, if R is small the current I may attain a very high value for a moderate value of the applied p. d. The consequence of this is that $\overline{V}_L$ and $\overline{V}_C$ may reach values which are enormously higher than the value of the applied p. d. $\overline{V}$.

Example.—Let $L=0{\cdot}03$ henry: $C=200$ micro-farads: $f=\frac{1}{2\pi\sqrt{LC}}=65$: $\omega=2\pi f=408$: $L\omega=\frac{1}{\omega C}$: $\overline{V}=100$ volts: $R=1$ ohm.

Then $\overline{I}=100$ ampères: $\overline{V}_L=L\omega\overline{I}=4980$ volts $=\overline{V}_C$.

The maximum value of the pressure across the inductance and across the capacity is thus about ten times greater than the maximum value of the applied p. d. across the whole circuit, and this high value of the pressure might be sufficient to rupture the insulation of the inductance and condenser.

The vector diagram for this case is shown in Fig. 308.

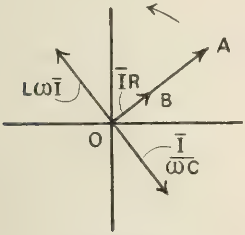


FIG. 308.

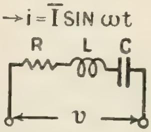


FIG. 309.

It is of interest to examine the conditions as the supply frequency is varied from a value well below resonance to a value considerably higher than resonance. Two cases will be considered—viz., one in which the resistance of the circuit is very low, and one in which the resistance is moderately high, the other constants of the circuit being the same in both cases.

In Fig. 309 let $L=0{\cdot}005$ henry: $C=200$ micro-farads: $\overline{V}=200$ volts.

Then
$$\overline{I}=\frac{\overline{V}}{\sqrt{R^2+\left(L\omega-\frac{1}{\omega C}\right)^2}}\text{ ampères,}$$
where
$$\omega=2\pi f.$$

The natural frequency of the circuit is $\frac{1}{2\pi\sqrt{LC}}=160$ cycles per second.

By assuming different values of the supply frequency f the corresponding value of $\overline{I}$ may be determined.

In Fig. 310 curve 1 shows the connection between the maximum value of the current wave $\bar{\mathrm{I}}$ and the frequency f when $\mathrm{R}=\frac{1}{4}$ ohm, and curve 2 gives the relationship when $\mathrm{R}=1$ ohm.

It is to be observed that for the smaller value of R the curve is very steep in the neighbourhood of resonance, whilst for $\mathrm{R}=1$ ohm the curve is much less steep and the top of the curve is relatively very flat.

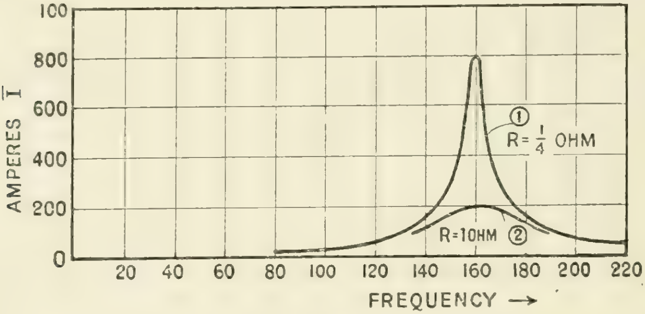


Fig. 310.

In Fig. 311 curve 1 shows the connection between the angles ϕ (in degrees) of phase displacement of the current on the applied p. d. and the supply frequency f when $R=\frac{1}{4}$ ohm. Curve 2 shows the relationship between the same two quantities when $\mathrm{R}=1$ ohm. The value of ϕ is given as before by the expression

$$\tan\phi=\frac{\mathrm{L}\omega-\dfrac{1}{\omega\mathrm{C}}}{\mathrm{R}}.$$

If the supply frequency is *less* than the resonant frequency the current *leads* on the applied p. d. If the supply frequency is *greater* than the resonant frequency the current *lags* on the applied p. d. It is also to be observed that a very small departure of the supply frequency from the resonant value causes the current to lag or lead by a relatively large angle when $\mathrm{R}=\frac{1}{4}$ ohm, whereas when $\mathrm{R}=1$ ohm the same variation of frequency causes a much smaller variation of the angle ϕ.

It will be clear from the foregoing that if, in a given circuit containing an inductance L and supplied with alternating current at a frequency f, it is possible to bring the current into phase with the applied p. d. by connecting in series with the circuit a condenser of suitable capacity, this is a result of great practical importance.

231. Circuits in Parallel.—In considering series arrangements of circuits the current is the factor which is the same throughout the circuit, and hence in determining the phase relationship of current and applied p. d. it is convenient to obtain an expression for the applied p. d. in terms of the current.

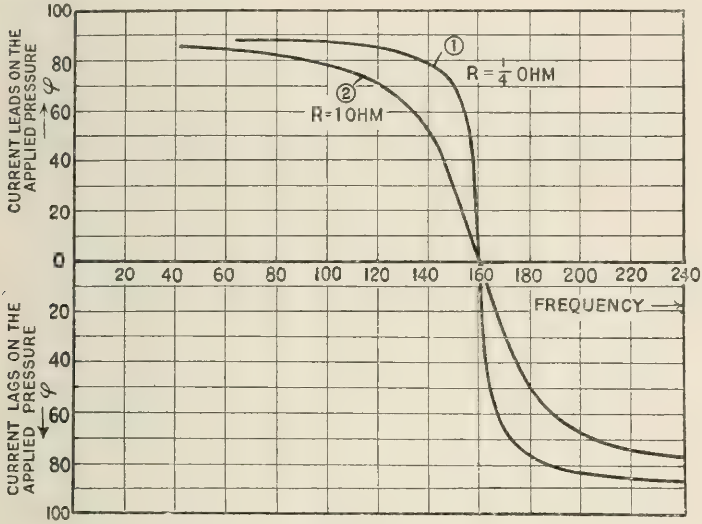


Fig. 311.

In the case of parallel circuits, such as *abc*, *adc* in Fig. 312, the applied p. d. is the same for each branch, and it is therefore convenient to use the p. d. as the basis of calculation.

Let the applied p. d. be—

$$v=\bar{V}\sin\omega t \text{ volts.}$$

The current in *abc* is—

$$i_1=\frac{\bar{V}}{\sqrt{R_1^2+(Lw)^2}}\sin(\omega t-\phi_1),$$

where $$\tan\phi_1=\frac{L\omega}{R_1}.$$

The current in *adc* is—

$$i_2=\frac{\bar{V}}{\sqrt{R_2^2+\left(\frac{1}{\omega C}\right)^2}}\sin(\omega t+\phi_2),$$

where $$\tan\phi_2=\frac{1}{\omega CR_2}.$$

The current in the main lead is therefore—

$$i=i_1+i_2$$

(§ 149, Section III.).

Hence—

$$i = V\left[\frac{1}{Z}\sin(\omega t-\phi_1)+\frac{1}{Z_2}\sin(\omega t+\phi_2)\right]\text{ampères,}$$

where $Z_1=\sqrt{R_1^2+(L\omega)^2}$ ohms; $Z_2=\sqrt{R_2^2+\left[\frac{1}{\omega C}\right]^2}$ ohms.

That is—

$$i=\bar{V}\left[\left(\frac{\cos\phi_1}{Z_1}+\frac{\cos\phi_2}{Z_2}\right)\sin\omega t-\left(\frac{\sin\phi_1}{Z_1}-\frac{\sin\phi_2}{Z_2}\right)\cos\omega t\right]$$

ampères

$$=\bar{V}\left[\left[\frac{R_1}{Z_1^2}+\frac{R_2}{Z_2^2}\right]\sin\omega t-\left(\frac{L\omega}{Z_1^2}-\frac{\frac{1}{\omega C}}{Z_2^2}\right)\cos\omega t\right]\text{ampères}$$

$$=V\sin(\omega t-\gamma)\sqrt{\left(\frac{R_1}{Z_1^2}+\frac{R_2}{Z_2^2}\right)^2+\left(\frac{L\omega}{Z_1^2}-\frac{\frac{1}{\omega C}}{Z_2^2}\right)^2}\text{ampères}$$

where
$$\tan\gamma=\frac{\dfrac{L\omega}{Z_1^2}-\dfrac{\frac{1}{\omega C}}{Z_2^2}}{\dfrac{R_1}{Z_1^2}+\dfrac{R_2}{Z_2^2}}$$

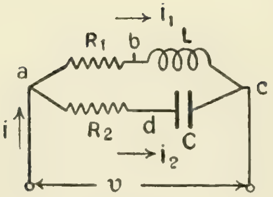


FIG. 312.

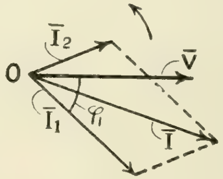


FIG. 313.

that is
$$\tan\gamma=\frac{\dfrac{Lw}{R_1^2+(Lw)^2}-\dfrac{\frac{1}{wC}}{R_2^2+\left(\frac{1}{wC}\right)^2}}{\dfrac{R_1}{R_1^2+(Lw)^2}+\dfrac{R^2}{R_2^2+\left(\frac{1}{wC}\right)^2}}.$$

The vector diagram for this case is given in Fig. 313.

If $L\omega = \frac{1}{\omega C}$—that is to say, the closed circuit—*abcda* is in resonance with the supply frequency—

$$\tan \gamma = \frac{(R_2 - R_1)\frac{1}{\omega C}}{R_1R_2 + \left(\frac{1}{\omega C}\right)^2} = \frac{(R_2 - R_1)\,L\omega}{R_1R_2 + (L\omega)^2}.$$

If $L\omega = \frac{1}{\omega C}$ and $R_2 = R_1$, $\tan \gamma = 0$, and the current is in phase with the applied p. d.

If $L\omega = \frac{1}{\omega C}$ and $R_2 = R_1 = 0$, then $i = 0$, and no current flows in the mains although currents flow in each branch—viz.:

$$i = o : i_1 = \frac{\overline{V}}{L\omega} \sin\left(\omega t - \frac{\pi}{2}\right) : i_2 = \overline{V}\omega C \sin\left(\omega t + \frac{\pi}{2}\right).$$

The closed circuit ABCDA acts in this case as a *perfect insulator.*

Similarly, in the case of a series circuit, if $L\omega = \frac{1}{\omega C}$ and $R_2 = R_1 = 0$, the circuit acts as a *perfect conductor.*

For another method of dealing with parallel circuits, see § 285, Chapter XXII.

232. Root Mean Square Values.—It was explained in § 145, Section III., that the power expended when a current flows through a resistance is proportional to the square of the current.

If the current flowing through a resistance of R ohms is i ampères the power expended is—

$$i^2R \text{ joules per second;}$$

that is, i^2R watts,

and this power is expended in heating the resistance.

Further, in any electrical apparatus the consuming device can always be considered as equivalent to a resistance, the power expended by the current being equal to the square of the current multiplied by the value of the equivalent resistance.

When the current is an alternating current the power varies from moment to moment, and the practical question arises as to what value has to be assigned to the current which, when squared and multiplied by the resistance, will give the mean value of the power expended in the resistance.

Let the current be—

$$i = \bar{I} \sin \omega t \text{ ampères.}$$

At any instant the power expended is—

$$i^2\mathrm{R} \text{ watts;}$$

that is, $$\mathrm{R}\bar{\mathrm{I}}^2 \sin^2\omega t \text{ watts.}$$

During one cycle the mean power is—

$$\frac{1}{\mathrm{T}}\int_0^{\mathrm{T}} \mathrm{R}\bar{\mathrm{I}}^2 \sin^2\omega t dt \text{ watts,}$$

where T is the time of one cycle—*i.e.*, $\frac{2\pi}{\omega}$ seconds.

Hence the mean power

$$=\frac{\bar{\mathrm{I}}^2\mathrm{R}}{\mathrm{T}}\int_0^{\mathrm{T}}\left(\frac{1-\cos\ 2\omega t}{2}\right)dt$$

$$=\frac{\bar{\mathrm{I}}^2\mathrm{R}}{2} \text{ watts.}$$

The equivalent or effective value of the current which, when squared and multiplied by the resistance, gives the mean power is therefore $\frac{\bar{\mathrm{I}}}{\sqrt{2}}$ ampères.

From the expression from which this effective value of the current is deduced it will be seen that this current is the square root of the mean square value of the current during one cycle. It is therefore called the **root mean square** (r. m. s.) value, or sometimes the **effective** value of the current, and will be denoted by I. Hence

$$\mathrm{I}=\frac{\bar{\mathrm{I}}}{\sqrt{2}}.$$

The r. m. s. value of a *sine wave* of current is therefore $\frac{1}{\sqrt{2}}$ times the maximum value. Similarly the r. m. s. value of a *sine wave* of e. m. f. or p. d. is $\frac{1}{\sqrt{2}}$ times the maximum value.

The r. m. s. value of an alternating current is clearly equal to that of a direct current which gives the same heating effect in a resistance.

There is another quantity which is of practical importance, chiefly in connection with the design of alternating current machines, and that is the *mean* value of a sine wave. The mean value of the sine wave $i=\bar{\mathrm{I}} \sin \omega t$ is—

$$\frac{2}{\mathrm{T}}\int_0^{\frac{\mathrm{T}}{2}} \bar{\mathrm{I}} \sin \omega t dt,$$

where $$\mathrm{T}=\frac{2\pi}{\omega}.$$

Hence the mean value is $$=\frac{2}{\pi}\,\mathrm{I}.$$

The ratio $\frac{\text{r. m. s. value}}{\text{mean value}}$ is termed the **form factor**, and for a sine wave is 1·11.

233. Power in Alternating Current Circuits—Power Factor.—The power expended in an alternating current circuit, or part of a circuit, at any moment is the product of the simultaneous instantaneous values for the circuit, or the part of the circuit considered of the current and applied p. d. (as in Section III., § 145).

Case I.—Suppose a sine wave of current flows in a non-inductive resistance of R ohms.

Let $$i=\bar{\mathrm{I}}\sin\omega t \text{ ampères.}$$

Then the applied p. d. at the terminals of the resistance will be—

$$v=\bar{\mathrm{V}}\sin\omega t=\bar{\mathrm{I}}\mathrm{R}\sin\omega t \text{ volts.}$$

The power at any moment t is—

$$vi=\mathrm{VI}\sin^2\omega t \text{ watts.}$$

The mean power is therefore—

$$\frac{1}{\mathrm{T}}\int_0^{\mathrm{T}}\bar{\mathrm{V}}\bar{\mathrm{I}}\sin^2\omega t\,dt \text{ watts}$$

$$=\frac{\bar{\mathrm{V}}\bar{\mathrm{I}}}{2}=\mathrm{VI} \text{ watts.}$$

That is, the mean power is the product of the r. m. s. value of the p. d. and the r. m. s. value of the current.

Case II.—Suppose a sine wave of current flows in a series circuit of non-inductive resistance R ohms, inductance L henrys, and capacity C farads.

Let $$i=\bar{\mathrm{I}}\sin\omega t \text{ ampères.}$$

Then $$v=\bar{\mathrm{V}}\sin(\omega t+\phi) \text{ volts,}$$

where $$\tan\phi=\frac{\mathrm{L}\omega-\frac{1}{\omega\mathrm{C}}}{\mathrm{R}}.$$

The power at any moment t is—

$$vi=\bar{\mathrm{V}}\bar{\mathrm{I}}\sin\omega t\sin(\omega t+\phi) \text{ watts.}$$

The mean power is—

$$\frac{1}{T}\int_{0}^{T}\bar{V}\bar{I}\sin\omega t\sin(\omega t+\phi)\,dt \text{ watts}$$

$$=\frac{\bar{V}\bar{I}}{T}\int_{0}^{T}(\sin^2\omega t\cos\phi+\sin\omega t\cos\omega t\sin\phi)dt \text{ watts}$$

$$=\bar{V}\bar{I}\left[\frac{\cos\phi}{2}\right] \text{ watts}$$

$$=\mathrm{VI}\cos\phi \text{ watts.}$$

Hence, the mean power is equal to the product of the r. m. s. value of the p. d. into the r. m. s. value of the current into the cosine of the angle of phase displacement between the current and p. d.

In a pure resistance circuit $\phi=0$: $\cos\phi=1$, and the mean power$=$VI.

The factor $\cos\phi$ is called the **power factor** of the circuit, and

$$\cos\phi=\frac{R}{\sqrt{R^2+\left(L\omega-\frac{1}{\omega C}\right)^2}}.$$

It is of interest to consider a little more closely the expression for the instantaneous value of the power—viz.:

$$w=vi=\bar{V}\bar{I}\sin\omega t\sin(\omega t+\phi) \text{ watts.}$$

This may be written—

$$w=\frac{\bar{V}\bar{I}}{2}[\cos\phi-\cos(2\omega t+\phi)] \text{ watts;}$$

that is— $w=\mathrm{VI}[\cos\phi-\cos(2\omega t+\phi)]$ watts.

The instantaneous value of the power therefore consists of two parts—viz.:

(i.) A *constant* quantity $\mathrm{VI}\cos\phi$,
(ii.) A quantity $\mathrm{VI}\cos(2\omega t+\phi)$.

The quantity (ii.) varies with the time with *twice the frequency* of the waves of current and p. d.

An example will make this clearer.

Example.—Let $i=\bar{I}\sin\omega t=30\sin 2\pi\times 50t$ ampères

$v=\bar{V}\sin(\omega t+\phi)=40\sin(2\pi\times 50t-35^\circ)$ volts.

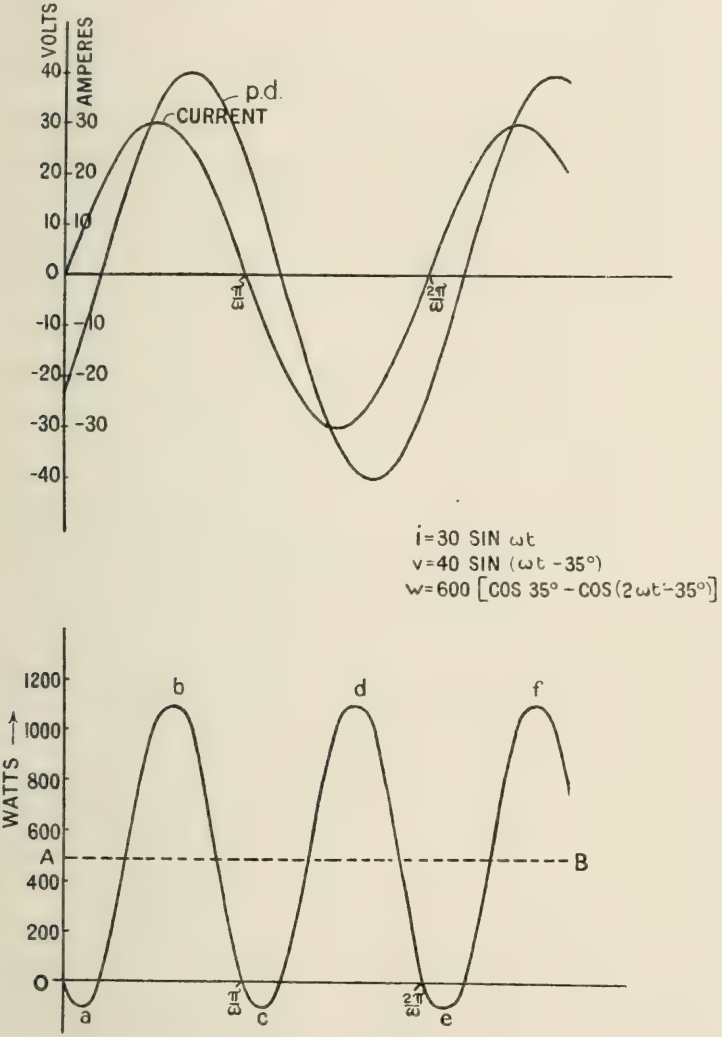
These waves of current and p. d. are shown in Fig. 314.

Then $w=vi=\bar{V}\bar{I}\sin\omega t\sin(\omega t+\phi)$ watts;

i.e., $w=600[\cos 35^\circ-\cos(2\pi\times 50t-35^\circ)]$ watts,

or $w=490-600\cos(2\pi\times 50t-35^\circ)$ watts.

This is shown in Fig. 315. The line AB represents the constant term of 490 watts, and the curve a*bcd* drawn on AB as abscissa gives the double frequency term. The mean value of



Figs. 314, 315.

the double frequency term over one period of the current is clearly zero, and hence the mean value of the power is 490 watts. The power factor in this case is cos 35°=0·82.

CHAPTER XVIII

SINGLE-PHASE, TWO-PHASE, AND THREE-PHASE ALTERNATING CURRENT SYSTEMS

234. Single-Phase System.—A single-phase system consists of a generator of alternating e. m. f., the armature of which has one single circuit, the terminals of the armature being connected to the consuming device (see Fig. 316). The generator e. m. f. is taken as positive when acting in the direction from A to A_1.

A diagrammatic sketch of a single phase a. c. generator is shown in Fig. 317. A single loop of wire is wrapped diametrically on an iron cylinder (armature) which revolves between the two magnet poles N and S. The ends are brought out each to one of two slip rings, and two brushes, one bearing on each slip ring, form the respective terminals of the armature.

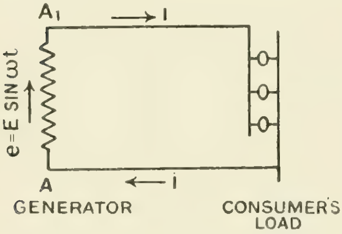


Fig. 316.

For simplicity of treatment it is assumed that the magnetic flux density in the gap between armature and pole faces is distributed according to the sine law. That is to say, if the armature periphery and pole faces be developed in a straight line the flux distribution will be as shown in Fig. 318.

In practice, however, the flux distribution curve is not a sine function, other methods being used to obtain an approximately sine wave form of e. m. f.

Let 11^1 represent the conductors of the loop, and suppose the armature revolves at a uniform speed of n revolutions per second in a counter-clockwise direction.

Let a cms. be the radius of the armature, and l cms. the axial length of the armature. The peripheral speed of the conductors will be—

$$v = 2\pi a n \text{ cms. per second.}$$

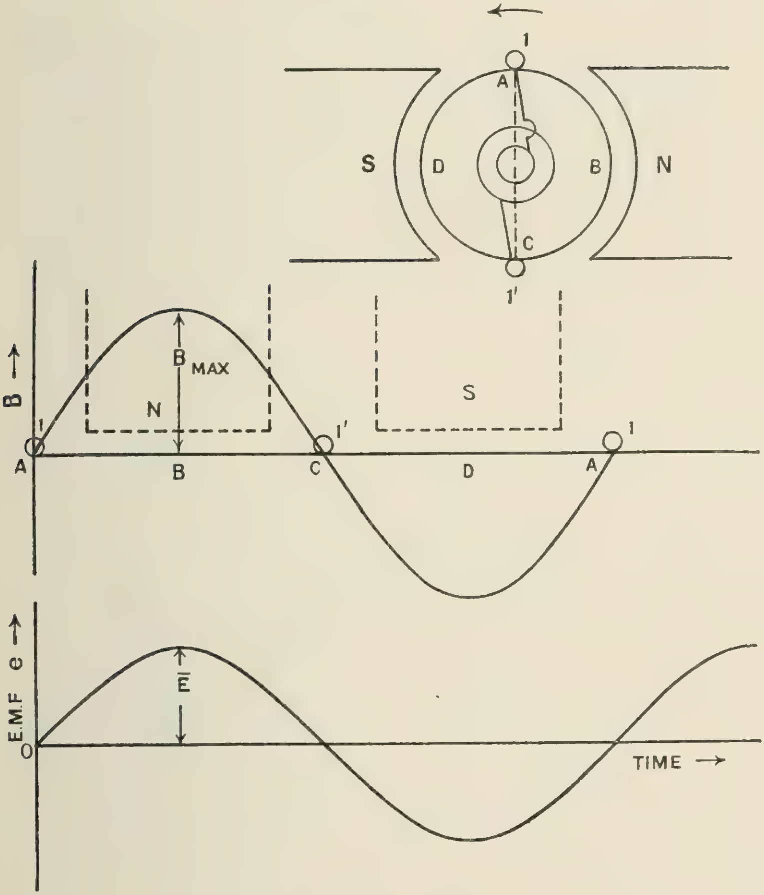
Let the flux density in the air gap be represented by the expression—

$$B_\theta = B_{max} \sin \theta,$$

where θ is the angle any point on the periphery of the armature makes with the vertical (Fig. 317).

Hence, at any time t the conductor 1 will be in a magnetic field of intensity—

$$B_{max} \sin \omega t,$$



Figs. 317, 318, 319.

where the angular velocity of the armature is—

$$\omega = 2\pi n,$$

and $$\omega t = \theta,$$

the zero of time being taken as the moment at which the conductor 1 has the position shown in Fig. 317—*i.e.*, $t=0$, when $\theta=0$.

The e. m. f. induced in the conductor 1 will therefore be (p. 273)—

$$e=\frac{vB_\theta l}{10^8} \text{ volts};$$

that is,
$$e=\frac{vl}{10^8} B_{max} \sin \omega t \text{ volts}.$$

The e. m. f. induced in conductor 1 when passing, say, under the S pole will be through the plane of the paper away from the observer for the given direction of rotation of the armature—*i.e.*, counter-clockwise rotation. The e. m. f. induced in conductor 1^1, when passing under the N pole, will be through the plane of the paper towards the observer for the same direction of rotation.

Hence the e. m. f.'s in both conductors are additive round the loop, and the total e. m. f. in the loop is—

$$e=\frac{2vl}{10^8} B_{max} \sin \omega t \text{ volts}.$$

If the coil has w turns—

$$e=\frac{2vwl}{10^8} B_{max} \sin \omega t \text{ volts};$$

or
$$e=\bar{E} \sin \omega t$$

where
$$\bar{E}=\frac{2vwl}{10^8} B_{max} \text{ volts},$$

and is the maximum value of the e. m. f. induced in the coil during the cycle.

In Fig. 319 the e. m. f. wave induced in the loop is shown as a function of the time.

The frequency of the e. m. f. is the number of complete cycles passed through in one second.

By reference to Figs. 317 and 319 it will be seen that in this case of a two-pole machine—

$$f=n=\frac{\omega}{2\pi} \text{ cycles per second}.$$

Example.—Let $B_{max}=10000$ lines per sq. cm.: $a=40$ cms.: $l=30$ cms.: $n=8$ revs. per sec.: $w=10$: $v=2\pi an=2000$ cms. per sec.: $\omega=2\pi n=50$.

Then
$$\bar{E}=120 \text{ volts}$$

and
$$e=120 \sin 50t \text{ volts}.$$

The r. m. s. value of this e. m. f. is $\frac{\bar{E}}{\sqrt{2}}=84$ volts.

The frequency is $f=\frac{\omega}{2\pi}=8$ cycles per second.

If the generator has p pairs of poles (see Fig. 320), then, in order that the e. m. f.'s in the two sides of the loop shall be additive, the sides of the loop are displaced along the periphery of the armature by the distance of two consecutive poles—*i.e.*, by $\frac{180}{p}$ degrees. It is usual to call the angle between two consecutive poles 180 *electrical degrees.* For a two-pole machine one electrical degree is equal to one geometrical degree, since $p=1$ for a two-pole machine. For a machine with p pairs of poles, one geometrical degree is equal to p electrical degrees.

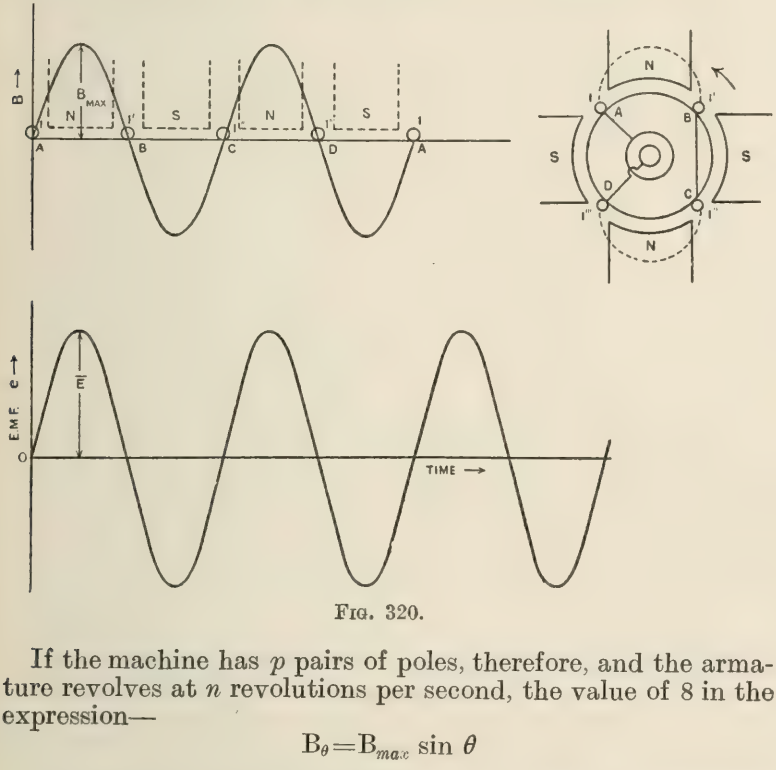


FIG. 320.

If the machine has p pairs of poles, therefore, and the armature revolves at n revolutions per second, the value of 8 in the expression—

$$B_\theta = B_{max} \sin \theta$$

is measured in electrical degrees.

The angular velocity ω is also to be taken in electrical measure—*i.e.*,

$$\omega = n2\pi p \text{ electrical radians,}$$

and $$\theta = \omega t.$$

The e. m. f. in each conductor is then—

$$\frac{vl\,B_{max} \sin \omega t}{10^8} \text{ volts, as before.}$$

If the coil is arranged, as in Fig. 320, so that there are $2p$ sides mutually separated by one pole pitch, the e. m. f. is—

$$e=\frac{2pvl\ \mathrm{B}_{max}}{10^8}\sin\omega t \text{ volts.}$$

If the coil has w turns—

$$e=\frac{2pwvl\overline{\mathrm{B}}}{10^8}\sin\omega t \text{ volts,}$$

or $$e=\overline{\mathrm{E}}\sin\omega t \text{ volts,}$$

where $$\overline{\mathrm{E}}=\frac{2pwvl\ \mathrm{B}_{max}}{10^8} \text{ volts.}$$

The frequency of the e. m. f. is the number of complete cycles passed through in one second. By reference to Fig. 320 it will be seen that—

$$f=np=\frac{\omega}{2\pi} \text{ cycles per second,}$$

ω being equal to $2\pi np$, as already stated.

Example.—Let $\mathrm{B}_{max}=10000$ lines per sq. cm.: $a=40$ cms.: $l=30$ cms.: $n=8$ revs. per sec.: $w=10$: $2p=4$: $\omega=\frac{2\pi np}{60}=100$: $v=2000$ cms. per sec.

Hence $$\overline{\mathrm{E}}=240 \text{ volts,}$$

and $$e=240\sin 100t \text{ volts.}$$

The r. m. s. value of the e. m. f. is—

$$\frac{240}{\sqrt{2}}=170 \text{ volts.}$$

The frequency $f=\frac{\omega}{2\pi}=16$ cycles per second.

235. Vector Diagram for a Single-Phase Transmission System.—Let $e=\overline{\mathrm{E}}\sin\omega t$ be the e. m. f. generated in the single-phase generator AA_1 (Fig. 316).

Let l henrys be the inductance and r ohms the resistance of the armature winding, and let L henrys be the inductance and R ohms the resistance of the load to which the generator is connected.

The current will be (§ 229)—

$$i=\overline{\mathrm{I}}\sin(\omega t-\phi_o),$$

where $$\tan\phi_o=\frac{\omega(\mathrm{L}+l)}{\mathrm{R}+r}.$$

In Fig. 321 OE is the vector of e. m. f. and OI the vector of current.

The drop of potential in the winding AA_1 is given by the vector OV_d,

where $og = \bar{I}r$ and $gV_d = l\omega I$.

The p. d. at the terminals of the winding is therefore a vector OV, such that—

$$OV + OV_d = OE,$$

and the angle of phase displacement of current relatively to this p. d. is ϕ—*i.e.*, $\cos\phi$ is the power factor of the load.

Suppose now that the constants L and R of the load vary, the p. d. OV (Fig. 321) being kept constant.

Draw a semicircle on OV as diameter (Fig. 322), and draw OA at an angle ϕ to OV where $\tan\phi = \frac{L\omega}{R}$.

Then the intercept OA will be $\bar{I}R$ and the chord AV will be $L\omega\bar{I}$, since the length of the vector OI represents $\bar{I}$ ampères.

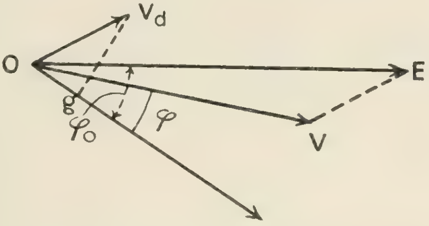


Fig. 321.

Similarly, if the constants $L\omega$ and R have any other values draw the line OA at the corresponding angle ϕ to intercept the semicircle at A.

The current vector OI will then coincide in direction with OA, and the magnitude of the current is given by the volts OA divided by the resistance R ohms.

236. Two-Phase Non-interlinked System.—A two-phase alternating current system is one having two circuits in which two separate e. m. f.'s act. The two e. m. f.'s are of the same frequency and of the same amplitude or maximum value, and are displaced in phase relatively to each other by one-quarter period.

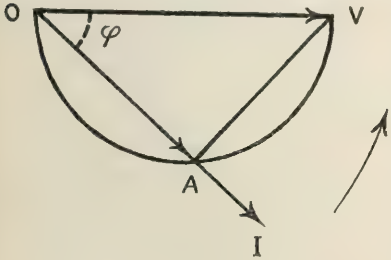


Fig. 322.

In Fig. 323 is shown in diagrammatic fashion an alternating current two-phase generator. Two separate coils are shown—viz., 11^1 and 22^1—each arranged diametrically on the armature of a two-pole machine, the planes of the coils being at right angles to each other. Each end of each coil is connected to a separate slip ring, and brushes bearing on the slip rings form the respective terminals of the armature winding. Such an arrangement of coils constitutes a two-phase, non-interlinked, armature winding, each coil forming one phase.

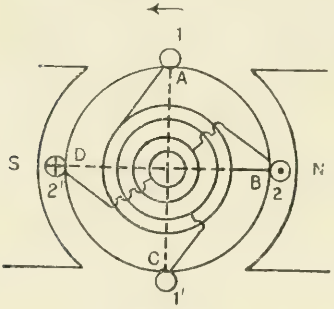
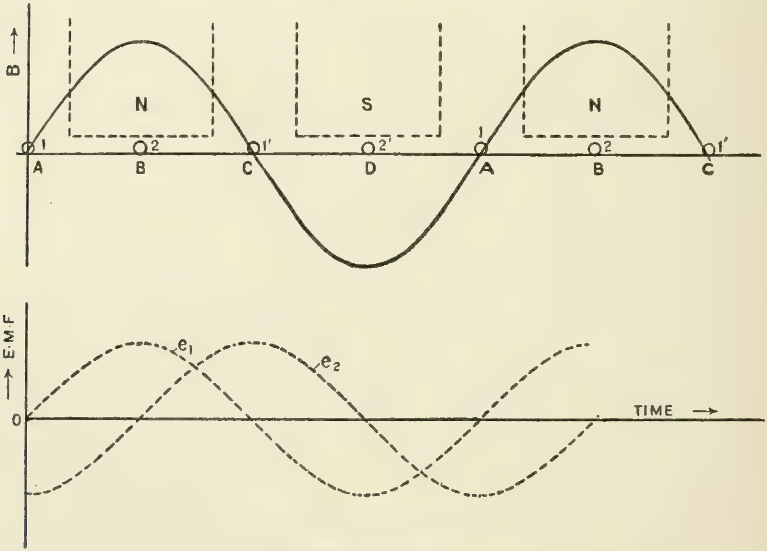


Fig. 323.

In Fig. 324 is shown the development in a straight line of the armature periphery and pole faces of Fig. 323. The curve of magnetic flux density at the periphery of the armature is drawn in Fig. 324, and it is assumed, for simplicity of treatment, that this flux density curve is of sine shape (as in § 234, Fig. 318).

Suppose the armature of Fig. 323 to rotate in a counter-clockwise direction—that is, the conductors 11^1 : 22^1 (Fig. 324) move from right to left. At the instant shown in Fig. 323 the e. m. f. induced in conductor 2 is directed through the plane of the paper towards the observer, and this is denoted by ⊙. At the same moment the e. m. f. induced in conductor 2^1 is directed through the plane of the paper away from the observer, and this is denoted by ⊕ (see p. 254, Section IV.). At every instant the e. m. f.'s induced in both conductors of a coil act in the same direction as each other round the coil, the total e. m. f. round the coil



Figs. 324, 325.

being twice the e. m. f. induced in either conductor of the coil. The e. m. f. induced in each coil will be a sine function of the time, as shown in Fig. 325, and the e. m. f.'s in the two coils will be displaced by one-quarter period relatively to each other. For the conditions illustrated in Figs. 323 and 324 the e. m. f. in coil 11^1 is one-quarter period ahead of the e. m. f. in coil 22^1.

Precisely as explained on p. 344, the e. m. f. induced in phase 11^1 is—

$$e_1 = \bar{E} \sin \omega t \text{ volts,}$$

where $\omega = 2\pi n$,

and $$\bar{E} = \frac{2vwl\ B_{max}}{10^8} \text{ volts,}$$

where w is the number of turns in the coil.

The zero of time is taken as the moment at which the conductor 1 has the position shown in Fig. 323.

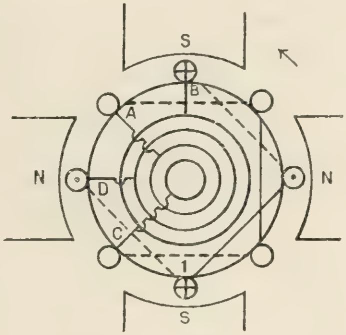


Fig. 326.

The e. m. f. induced in phase 22^1 is one-quarter period behind that in 11^1.

Hence $$e_2 = \bar{E} \sin (\omega t - 90^\circ) \text{ volts,}$$

where E has the same value as for e_1.

The e. m. f. waves e_1 and e_2 are shown in Fig. 325.

The frequency of the e. m. f. in each phase is—

$$f = n = \frac{\omega}{2\pi} \text{ cycles per second.}$$

If the generator has p pairs of poles the conductors of each phase may be arranged, as shown in Fig. 326. Any two consecutive conductors of a phase are displaced by the distance of one pair of poles —*i.e.*, by $\frac{180}{p}$ geometrical degrees or 180 *electrical degrees* (see p. 345). The value of ω in the above formula is to be taken in electrical degrees—that is

$$\omega = 2\pi np \text{ (as in p. 345).}$$

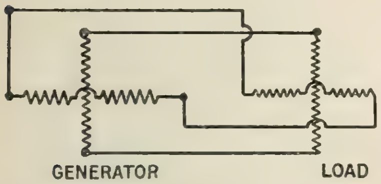


Fig. 327.

The amplitude of the e. m. f. induced in each phase is—

$$\bar{E} = \frac{2pwvl\ B_{max}}{10^8} \text{ volts (as in § 234).}$$

The frequency of the e. m. f. is—

$$f=np=\frac{\omega}{2\pi} \text{ cycles per second.}$$

The diagrammatic representation of such a two-phase non-interlinked system is given in Fig. 327. Such a system may thus be considered as two separate independent single-phase systems. If the current and power factor of each phase is the same, the system is said to be *balanced.*

237. Two-Phase Interlinked System.—A very common arrangement of a two-phase system is one in which one end of each phase is connected to a common slip ring of the armature, so that only three slip rings are necessary, and consequently only three wires lead from the armature, as shown diagrammatically in Fig. 328. The wire OC is common to both phases. This arrangement is known as a two-phase *interlinked* or three-wire system.

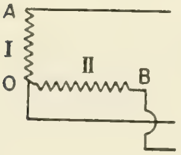


FIG. 328.

In Fig. 329 let OE_I be the e. m. f. induced in phase I, and OE_{II} the e. m. f. induced in phase II. As regards the p. d. between the terminals A and B (Fig. 328), it will be useful to consider a simple preliminary example.

Suppose the three points O, A, and B in Fig. 330 are connected to a battery so that the potential of O is zero, the potential of A is +20 volts, and that of B +30 volts. The potential of B with respect to A is $30-20=+10$ volts. The potential of A with respect to B is $20-30=-10$ volts. That is to say, the potential of A with respect to B is obtained by reversing the sign of B's potential and adding to the potential of A.

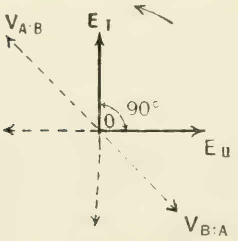


FIG. 329.

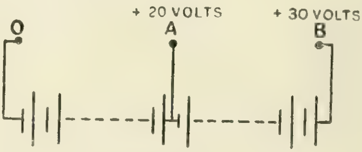


FIG. 330.

In the case of alternating current problems a precisely similar method is used. Thus, in Fig. 329, to find the potential of A with respect to B the vector OE_{II} (Fig. 329) is reversed and added to the vector OE_I, thus giving the vector $V_{A:B}$. The same result is, of course, obtained by joining $E_{II}E_I$. Similarly the potential of B with respect to A is $V_{B:A}$ in Fig. 329.

Dec [illegible]

The p. d. $V_{A:B}$ or $V_{B:A}$ is known as the *line pressure*, as distinguished from the p. d.'s OE_I and OE_{II}, which are termed the *phase pressures.* It will be clear from Fig. 329 that the line pressure of a two-phase generator is $\sqrt{2}$ times the phase pressure.

238. E. M. F., Current, and P. D. Vector Diagram for a Two-Phase, Interlinked, Balanced System.—Since the system is balanced (§ 236), the current and power factor of the load on each phase of the generator is the same (Fig. 333).

Let $l\omega = x$ ohms be the reactance and r ohms the resistance of each phase of the generator winding.

Let $L\omega = X$ ohms be the reactance and R ohms the resistance per phase of the load. In the first place, assume that the reactance and resistance of all the connecting cables is zero—*i.e.*, $x_e = 0 : r_e = 0$ (see Fig. 331).

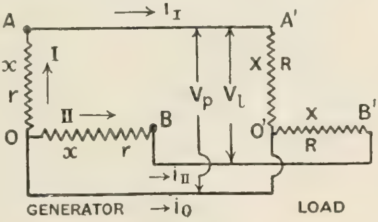


Fig. 331.

The positive direction of the e. m. f.'s and currents in the generator windings is away from O and in the load is towards O^1.

Let the e. m. f. in phases I and II be respectively—

$$e_I = \bar{E} \sin \omega t : e_{II} = \bar{E} \sin\left(\omega t - \frac{\pi}{2}\right).$$

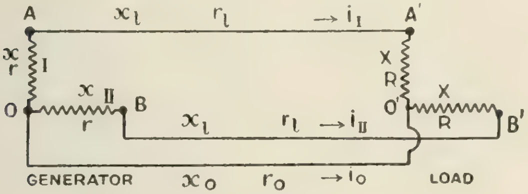


Fig. 332.

The total reactance of the circuit of phase I—*i.e.*, the circuit OAA^1O^1O—is $(X+x)$ ohms, and the total resistance $(R+r)$ ohms. Hence the current in phase I is—

$$i_I = \frac{\bar{E}}{\sqrt{(X+x)^2+(R+r)^2}} \sin(\omega t - \phi_o) \text{ ampères,}$$

where $$\tan \phi_o = \frac{X+x}{R+r};$$

i.e., $$i_I = I \sin(\omega t - \phi_o) \text{ ampères,}$$

where $$\bar{I} = \frac{\bar{E}}{\sqrt{(X+x)^2+(R+r)^2}} \text{ ampères.}$$

Similarly $i_{\text{II}} = \bar{\text{I}} \sin (\omega t - 90^\circ - \phi_\circ)$ ampères

Now suppose that the reactance and the resistance of the third wire are respectively $x_\circ$ and $r_\circ$, and let x_l and r_l be the reactance and resistance respectively of each cable AA^1 and BB^1 (Fig. 332).

In Fig. 333, OP$_1$ is the pressure necessary to overcome the reactance and resistance of the phase OA of the generator and the line AA^1, the current being represented by the vector O_{I_1}. The vector OV$_1$ is the pressure of the load terminal A^1 with respect to the point O in the generator, and is such that the e. m. f. vector OE$_1$ is the sum of the vector OP$_1$ and OV$_1$. Similarly for the vectors OP$_{\text{II}}$, OV$_{\text{II}}$, and OE$_{\text{II}}$ for phase II, the current in this phase being represented by the vector OI$_{\text{II}}$. The current in the line OO^1 is given by the vector OI$_\circ$, and is such that the vector sum of OI$_{\text{I}}$, OI$_{\text{II}}$, and OI$_\circ$ is zero.

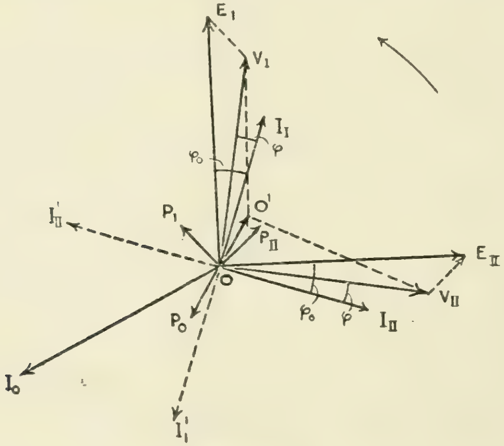


Fig. 333.

The vector OP$_\circ$ gives the pressure necessary to send the current through the line OO^1 (Fig. 332). The pressure of A^1 with respect to O^1 will be given by the vector difference of OV$_{\text{I}}$ and OP$_\circ$—that is, by O^{1}V$_{\text{I}}$ where OO^1 is equal to OP$_\circ$. Similarly, O^{1}V$_{\text{II}}$ gives the pressure of B^1 with respect to O^1.

It is thus seen that the phase pressures at the consumer's terminals are not symmetrical when the line OO^1 (Fig. 332) has appreciable impedance, although the current and the angles $\phi_\circ$ and ϕ for each phase is the same.

From Fig. 333 it is seen that the amplitude of OI$_\circ$ is—

$$\sqrt{2}\,\text{OI}_{\text{I}} = 1{\cdot}41\,\text{OI}_{\text{I}}.$$

Hence for the same current density the cross-sectional area of the cable OO^1 need only be 1·41 times the section of each of the cables AA^1 and BB^1. Hence, by using a three-wire (*i.e.*, interlinked) two-phase system an appreciable economy of copper is obtained as compared with a four-wire (*i.e.*, non-interlinked) two-phase system. In a four-wire system the cross-sectional area of the four cables is 4A where A is the section of each cable. In a three-wire system the sectional area of the three cables is $2A + 1{\cdot}41A = 3{\cdot}41A$. The saving in copper is thus—

$$\frac{4 - 3{\cdot}41}{4} = 15 \text{ per cent.}$$

To summarise the results for a two-phase interlinked system:

The line pressure has a magnitude $\sqrt{2}$ times that of the phase pressure, and the magnitude of the current in the third wire $\sqrt{2}$ times that of the current in each line.

If the third wire has appreciable impedance, however, reference to Fig. 333 shows that the line pressure *at the consumer's terminals* is not necessarily $\sqrt{2}$ times the phase pressure.

239. Three-Phase System.—A three-phase alternating current system is one in which three separate e. m. f.'s of the same frequency act. The e. m. f.'s are mutually displaced in phase by one-third period and are all of equal magnitude if the system is *symmetrical* If the phase displacement between the e. m. f.'s of any two phases is not one-third period, or if the magnitudes of the e. m. f.'s are not all equal, the system is *unsymmetrical.* A three-phase system is said to be *balanced* if the load on each phase is exactly the same as regards current, power factor, and phase p. d.

A three-phase system which is symmetrical on open circuit remains symmetrical when loaded with a balanced load.

The phase windings of a three-phase system may be connected in one of two ways—viz., *star* or *mesh* (*i.e.*, *delta*).

240. Star-Connected Three-Phase Generator.—In Fig. 334 three coils—11^1, 22^1, 33^1—are shown wound diametrically on the armature of a two-pole machine. The coils are mutually displaced on the armature by 120°—viz., the distance between the conductor 1 and 2 is one-third of the periphery of the armature, as is also the distance between the conductors 2 and 3. The ends of the conductors 1, 2, and 3 are brought out each to a separate slip ring on the armature shaft. The ends of the conductors 1^1, 2^1, and 3^1 are connected together to form a common junction known as the *neutral point* of the winding. This common junction is sometimes connected to a fourth slip ring on the armature shaft. If the armature (Fig. 334) rotate in a counter-clockwise direction the relative phase displacements of the waves of e. m. f. induced in coils 11^1, 22^1, 33^1 will be as shown by I, II, III respectively in Fig. 356, on the assumption that the magnetic flux density at the surface of the armature is distributed according to a sine law.

A diagrammatical representation of a star-connected generator is shown in Fig. 335, the neutral point being denoted by O. The e. m. f. in each phase is taken as being positive when directed from O outwards.

If the e. m. f. of phase I is denoted by—

$$e_{\mathrm{I}}=\overline{\mathrm{E}} \sin \omega t,$$

this will correspond to the zero of time being taken as the moment at which the conductor 1 is in the position shown in Fig. 334.

The e. m. f.'s in phases II and III will be given respectively by—

$$e_{\mathrm{II}}=\overline{\mathrm{E}} \sin \left(\omega t-\frac{2\pi}{3}\right) : e_{\mathrm{III}}=\overline{\mathrm{E}} \sin \left(\omega t-\frac{4\pi}{3}\right).$$

These e. m. f.'s are shown in the vector diagram of Fig. 336.

The e. m. f. between any two lines—*e.g.*, the potential of A (Fig. 335) with respect to B ($\mathrm{E}_{\mathrm{I:II}}$ in Fig. 336)—is found by reversing the vector $\mathrm{OE}_{\mathrm{II}}$ and adding to the vector OE_{I} (§ 237); or, what is the same thing, by drawing the vector $\mathrm{E}_{\mathrm{II}}\mathrm{E}_{\mathrm{I}}$. Similarly for the potential of B with respect to C and of C with respect to A.

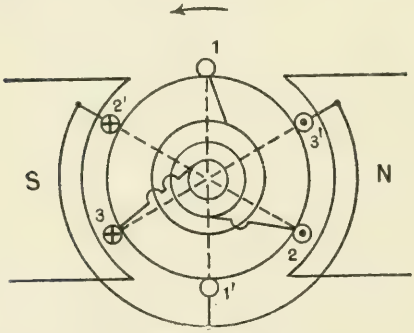


FIG. 334.

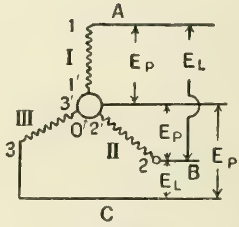


FIG. 335.

From Fig. 336 it will be seen that the amplitude of the line e. m. f. of the star-connected generator is $\sqrt{3}$ times the amplitude of the phase e. m. f., or

$$\overline{\mathrm{E}}_l=\sqrt{3}\ \overline{\mathrm{E}}_p.$$

241. Mesh-Connected Three-Phase Generator.—In Fig. 337 three coils—11^1, 22^1, 33^1—are shown arranged on the armature of a two-pole machine in a precisely similar fashion to those shown in Fig. 334. The coils in Fig. 337, however, are connected so that the beginning of coil 11^1 is connected to the end of coil 22^1, the beginning of coil 22^1 to the end of coil 33^1, and the beginning of coil 33^1 to the end of coil 11^1—that is to say, conductors 1 and 2^1, 2 and 3^1, 3 and 1^1 are connected together. Each junction is then joined to one of three slip rings.

For a mesh-connected armature the e. m. f. between any two lines is obviously the same as the e. m. f. in the phase winding connecting the two lines, or—

$$\bar{E}_l = \bar{E}_p.$$

If the e. m. f.'s in the respective phases be represented by

$$e_{I} = \bar{E} \sin \omega t : e_{II} = \bar{E} \sin \left(\omega t - \frac{2\pi}{3}\right) : e_{III} = \bar{E} \sin \left(\omega t - \frac{4\pi}{3}\right),$$

the sum of the e. m. f.'s round the mesh at that instant is—

$$e_{I} + e_{II} + e_{III} = \bar{E}\left[\sin \omega t + \sin\left(\omega t - \frac{2\pi}{3}\right) + \sin\left(\omega t - \frac{4\pi}{3}\right)\right] = 0.$$

That is, the e. m. f. round the mesh is zero at every instant, and consequently no current will flow in the mesh when the machine is on open circuit.

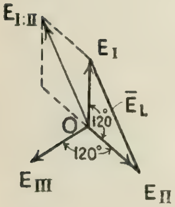


FIG. 336.

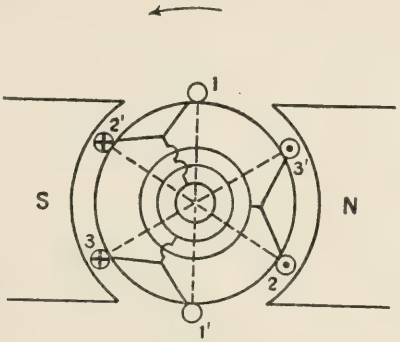


FIG. 337.

In practice, three-phase machines are built with more than two poles, and the coil sides (*e.g.*, 1 and 2, Fig. 334) are then displaced by $\frac{2}{3}$ pole pitch or $\frac{2\pi}{3}$ radians in electrical measure, as explained in § 234.

242. Star-Connected and Mesh-Connected Balanced Loads.—The load may be connected in either star or mesh, and supplied by either a star or mesh wound generator.

Case I. Star-Connected Load and Star-Connected Generator.—This arrangement is shown in Fig. 338. As has been stated in § 240, the positive direction of e. m. f.'s in a star-wound generator is assumed to be *away from* the neutral point. In a star-connected load, however, the positive direction is assumed to be *towards* the neutral point, as shown in Fig. 338.

It is clear that the current in each phase of the load is the same as that in the corresponding phase of the generator and connecting line.

That is— $\bar{I}_p = \bar{I}_l.$

The current in the line OO^1 joining the two neutral points is—

$$i_{\rm I}+i_{\rm II}+i_{\rm III}=\bar{\rm I}\sin(\omega t-\phi)+\bar{\rm I}\sin\left(\omega t-\phi-\frac{2\pi}{3}\right)+\bar{\rm I}\sin\left(\omega t-\phi-\frac{4\pi}{3}\right)$$
$$=0.$$

Hence, the conductor OO^1 may be omitted without disturbing the conditions.

If r and x be respectively the resistance and reactance per phase of the generator, and R and X respectively the resistance and reactance per phase of the load.

The current in each phase will lag behind the e. m. f. by an angle ϕ_0, such that—

$$\tan\phi_0=\frac{X+x}{R+r},$$

as shown in Fig. 339.

In the vector diagram of Fig. 340, OP_1 represents the pressure necessary to overcome the impedance in phase I of the armature winding due to the current OI_1, and OV_1 is the resultant p. d. between terminal A and the neutral point O. The vector sum of OP_1 and OV_1 is the induced e. m. f. OE_1. Similarly for the other two phases.

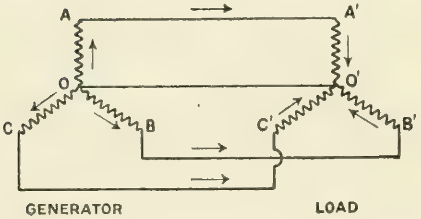


Fig. 338.

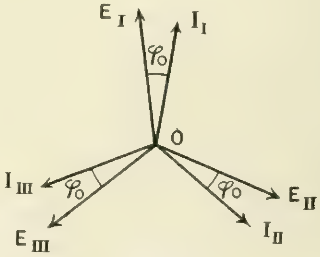


Fig. 339.

The angle ϕ (Fig. 340) is thus the angle of lag of the phase current on the terminal phase pressure of the machine. The potential of the line A with respect to the line B (Fig. 338) is given by a vector, such as OV_1V_2 in Fig. 362, and similarly for the pressure of line B with respect to line C, and of line C with respect to line A. The pressure between the lines is therefore $\sqrt{3}$ times the pressure per phase.

Case II. Mesh-Connected Load and Mesh-Connected Generator.—This arrangement is shown in Fig. 341. Since the load is balanced, considerations of symmetry show that the current in each phase of the load mesh is precisely the same as that in each phase of the generator mesh.

For the diagrammatical representation of a mesh-connected generator, as shown in Fig. 341, the positive direction of e. m. f.'s and current is taken to be counter-clockwise, and for the mesh-connected load the positive direction is clockwise.

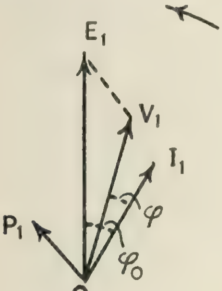


FIG. 340.

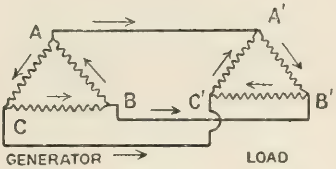


FIG. 341.

In Fig. 342 the vector diagram of e. m. f.'s for the generator is shown, and also the current vectors for each phase of the winding, the angle of lag of the current on the e. m. f. being ϕ_0. It will be clear that the sum of the currents round the mesh is zero at every instant, since the algebraical sum of the projection of the current vectors on the vertical is always zero.

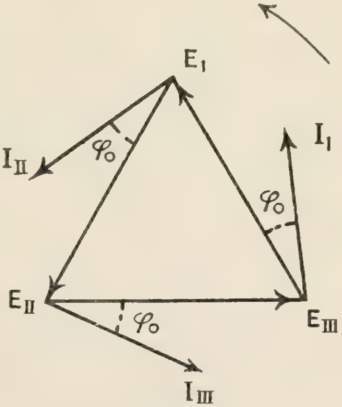


FIG. 342.

The current in any line, say AA^1 (Fig. 341), is such that at any instant the sum of the currents flowing toward the junction A is equal to the sum of the currents flowing away from this junction (§ 149). Hence, in Fig. 341 at any instant

current in AA^1 = current in BA
− current in AC,

or the line current $\bar{I}_l$ in AA^1 is given by the vector difference of the currents in phases BA and AC—

that is,
$$\bar{I}_l = \sqrt{3}\, \bar{I}_p.$$

The line current is therefore $\sqrt{3}$ times the phase current in a three-phase symmetrical balanced mesh connection.

The vector diagram of Fig. 340 refers to one phase of the mesh generator as well as to one phase of a star generator, OI_1 being the vector of phase current, OV_1 being the vector of phase p. d., and OP_1 the vector of phase pressure necessary to overcome the impedance of the phase.

In the case of a mesh generator the line pressure is obviously the same as the phase pressure.

Case III. Mesh-Connected Load and Star-Connected Generator.—This arrangement is shown in Fig. 343. From what has been said in reference to Case II. it will be clear that the line current is equal to $\sqrt{3}$ times the phase current in the load, and is the same as the phase current in the generator. The line pressure is the same as the phase pressure in the load, and is equal to $\sqrt{3}$ times the phase pressure in the generator.

Case IV. Star-Connected Load and Mesh-Connected Generator.—The relationship of phase and line currents and e. m. f.'s may be immediately deduced from the results given for Case III. (Fig. 343).

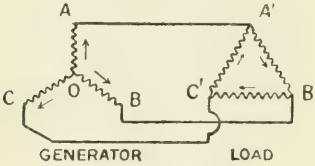


FIG. 343.

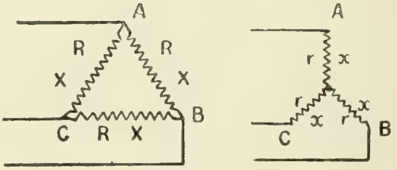


FIG. 344.

243. Transformation of a Balanced Mesh-Connected Load into an Equivalent Balanced Star-Connected Load.—In Fig. 344 let R, X, and Z be respectively the resistance, reactance, and impedance per phase of the mesh, and r, x, and z similar quantities per phase of the star.

Then $$Z=\sqrt{R^2+X^2} : z=\sqrt{r^2+x^2}.$$

In order that the two arrangements shall be equivalent, the resistance, reactance, and impedance between any two lines, say AC, in the mesh must be the same as the resistance, reactance, and impedance between the same two lines AC in the equivalent star.

To find the impedance of the two parallel branches between A and C in the mesh and the series arrangement between A and C in the star.

Let $v=\overline{V}\sin\omega t$ volts be the applied p. d. between the lines AC in the two cases (Figs. 345 and 346).

The current in the branch AC (Fig. 345) is—

$$i_1=\frac{\overline{V}}{\sqrt{R^2+X^2}}\sin(\omega t-\phi)=\frac{\overline{V}}{Z}\sin(\omega t-\phi).$$

The current in the branch ABC (Fig. 345) is—

$$i_2 = \frac{\bar{V}}{2\sqrt{R^2+X^2}} \sin(\omega t - \phi) = \frac{\bar{V}}{2Z} \sin(\omega t - \phi).$$

The current in the lines connected to A and C respectively (Fig. 345) is—

$$i = i_1 + i_2 = \frac{3}{2}\frac{\bar{V}}{Z} \sin(\omega t - \phi).$$

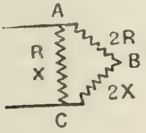


Fig. 345.

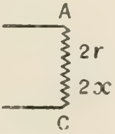


Fig. 346.

Hence the resultant impedance of the two parallel branches in Fig. 345 is—

$$\frac{2}{3} Z.$$

The impedance of the series arrangement in Fig. 346 is obviously $2\sqrt{r^2+x^2}$—*i.e.*, $2z$.

For the two arrangements (Figs. 345 and 346) to be identical, the magnitudes of the impedances must be equal—that is,

$$2z = \frac{2}{3} Z,$$

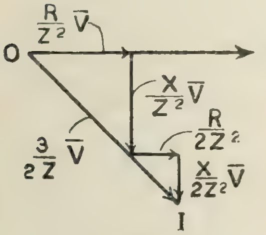


Fig. 347.

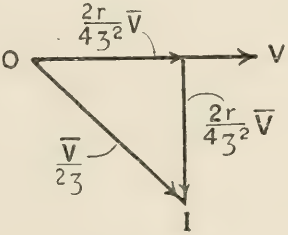


Fig. 348.

and the power factors of the two arrangements must also be equal—that is,

$$2r = \frac{2}{3}R : 2x = \frac{2}{3}X.$$

Hence $$Z = 3z : R = 3r : X = 3x.$$

The same result is arrived at by drawing the vector diagrams. Fig. 347 is the vector diagram for the parallel branches of Fig. 345, the vector OI being the resultant line current. Fig. 348 is the vector diagram for the arrangement of Fig. 346. The vectors OV and OI in both these diagrams are of the same magnitude and relatively displaced by the same phase angle. Inspection of Figs. 347 and 348 shows that—

$$Z = 3z : R = 3r : X = 3x, \text{ as before.}$$

Further reference to this problem will be found in § 288, Chapter XXII.

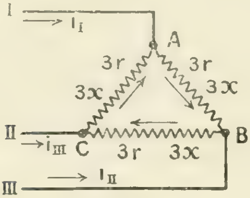


FIG. 349.

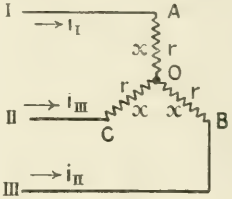


FIG. 350.

In Figs. 349 and 350 are shown equivalent mesh and star loads—that is to say, these two arrangements are identical, as regards the line currents and power factor of the load, when connected to a three-phase supply.

244. Vector Diagram showing Phase Relationships of Current and Pressure in a Mesh-Connected Load and in the Equivalent Star-Connected Load respectively.—The equivalent mesh and star-connected loads are shown in Figs. 349 and 350 respectively. In Fig. 351 the vector AB gives the pressure of A with respect to B, and AI$_1$ is the vector of current in the phase AB (Fig. 349) of the mesh connection. Similarly for the other phases.

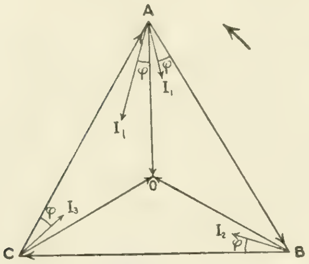


FIG. 351.

The vector AO gives the pressure of A with respect to O of the phase AO (Fig. 350 of the star connection), and similarly for the other phases.

The vector of current in the line connected to A (Fig. 349) is obtained by subtracting the vector of current in phase CA from the vector of current in phase AB (see p. 397)—that is, the vector of line current in the line connected to A (Fig. 349) is given by AI$_l$ (Fig. 351), which is the vector difference of AI$_1$ and CI$_3$, and the line current in Fig. 349 is clearly the phase current in the equivalent star connection of Fig. 350.

It is thus seen that the angle ϕ by which the phase current is displaced on the phase pressure in the mesh connection is the same as the angle by which the phase current is displaced on the phase pressure in the equivalent star connection.

245. Ring Winding.—A different method of winding the armature of an alternating current generator is the "closed circuit" winding of which the ring winding is a typical example. In Fig. 352 a simple example of such a winding is shown. The armature is an annular ring of iron, and the conductors are equally spaced and are wound on in a continuous spiral. The magnetic flux enters the semi-periphery of the ring facing the N pole, and leaves by the semi-periphery facing the S pole. The parts of the coil which lie on the outer periphery are the only ones which cut the flux, and consequently the only ones in which e. m. f.'s are induced. The parts of the coils lying on the inner periphery and also the ends of the coils act as connectors joining in series the active conductors. For the counter-clockwise direction of rotation, shown in Fig. 352, the e. m. f.'s in the conductors facing the N pole will be through the plane of the paper towards the observer, as denoted by the sign ⊙, and the e. m. f.'s in the conductors facing the S pole will be through the plane of the paper away from the observer, as denoted by the sign ⊕.

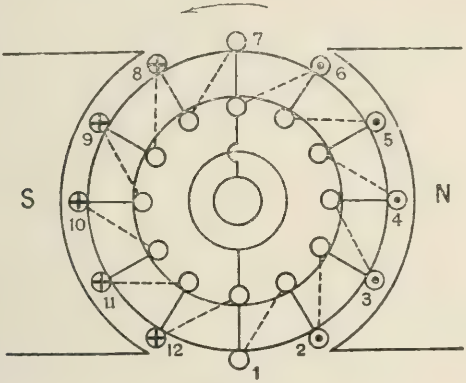


FIG. 352.

It will be seen that the resultant e. m. f. round the ring will be zero, since the e. m. f.'s in one-half of the conductors balance the e. m. f.'s in the other half.

246. Single-Phase Ring Winding.—Suppose two diametrically opposite conductors—*e.g.*, 1 and 7 (Fig. 352)—be connected one to each of two slip-rings. There will be an alternating e. m. f. acting between the slip-rings.

Suppose the distribution of the flux density on the periphery of the armature is a sine curve, and assume the zero of time to be the moment when the conductor 1 is in the position shown in Fig. 352. If there are twelve active conductors the e. m. f.'s in any two consecutive conductors will be displaced in phase by $\frac{2\pi}{12}$ radians.

Hence

$$e_1 = \mathrm{E} \sin(\omega t).$$

$$e_2 = \mathrm{E} \sin\left(\omega t + \frac{\pi}{16}\right).$$

$$e_3 = \mathrm{E} \sin\left(\omega t + \frac{\pi}{3}\right).$$

In Fig. 353 the waves of e. m. f. induced in conductors 1 to 6 are plotted in the correct phase relationship. The resultant e. m. f. wave is obtained by adding corresponding ordinates of the six individual waves, as shown by e_R in Fig. 353.

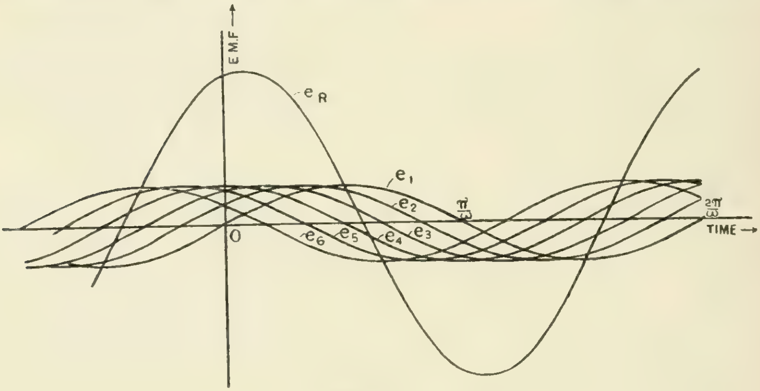


Fig. 353.

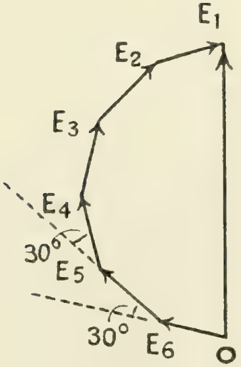


Fig. 354.

The vector diagram for such a single-phase ring winding is shown in Fig. 354, the e. m. f.'s in the six conductors (1 to 6) being represented respectively by the vectors $E_2E_1 : E_3E_2$ . . . OE_6, any two consecutive vectors being inclined at an angle of 30° to each other, the direction of rotation of the vectors being counter-clockwise, so that OE_6 is $\frac{1}{12}$ period ahead of E_6E_5, and so on.

The resultant e. m. f. which acts between the slip-ring in Fig. 352 is thus given by the vector OE_1, and this vector corresponds to the amplitude of the wave e_R in Fig. 353.*

---

* Figs. 353 and 354 have been reproduced to different scales. If the individual vectors OE_6, etc., were drawn of length equal to the amplitudes of the individual waves e_6, etc. (Fig. 353), the length of the resultant vector OE_1 would have been equal to the amplitude of the resultant wave e_R.

The vector of e. m. f. induced between the slip-ring due to the conductors 7 to 12 is also given by OE_1, and the armature winding thus consists of two parallel branches in each of which an e. m. f. of the same magnitude is induced, and acting in the same direction with respect to the slip-ring and the external circuit. The resultant e. m. f. for the whole internal circuit of the ring is, however, zero, and therefore no current will flow in the ring when the external circuit is open.

It is to be observed with reference to Fig. 354 that the amplitude of the resultant e. m. f. OE_1 is less than the arithmetical sum of the amplitude of the individual e. m. f.'s, OE_6, E_6E_5, etc.

If $\overline{E}$ is the amplitude of the e. m. f. induced in each conductor —*i.e.*, the length of the individual vectors OE_6, etc., in Fig. 354— the amplitude of the resultant e. m. f. OE_1 is easily deduced from Fig. 354.

Thus, if a circle be described on OE_1 as diameter, the ends of the vectors OE_6, E_6E_5, etc., will lie on the circumference.

Hence
$$\frac{OE_1}{2}\sin 15^\circ=\frac{OE_6}{2}=\frac{\overline{E}}{2},$$

or
$$OE_1=\frac{\overline{E}}{\sin 15^\circ}=\frac{\overline{E}}{0{\cdot}259}=3{\cdot}86\ \overline{E}.$$

The ratio
$$\frac{OE_1}{OE_6+E_6E_5+\ .\ .\ .\ E_2E_1}$$
is called the *winding factor* of this single-phase winding, and in this case is—
$$\frac{OE_1}{6E}=\frac{3{\cdot}86}{6}=0{\cdot}643.$$

If instead of six conductors in series between the slip-rings there were a very large number of conductors in series, the vector diagram $OE_6E_5\ .\ .\ .\ E_1$ would merge into a semicircle on the diameter OE_1, and the winding factor would be—
$$\frac{OE_1}{\frac{\pi}{2}\,OE_1}=\frac{2}{\pi}=0{\cdot}636.$$

For a tabulated list of winding factors reference should be made to pp. 407 and 408.

247. Three-Phase Ring Winding.—Now suppose that the winding in Fig. 355 is tapped at three points at 120° apart, and each tapping point connected to a separate slip-ring. The e. m. f. induced, for example, between the inner and outer slip-rings of Fig. 355 is the resultant of the e. m. f.'s induced in the conductors 1, 2, 3, and 4.

In Fig. 356 the curves of the e. m. f.'s induced in these conductors 1, 2, 3, and 4 are shown in their correct phase relationship—*i.e.*, each displaced from the neighbouring one by $\frac{1}{12}$ period. The resultant e. m. f. is obtained by adding corresponding ordinates, and is shown by wave e_R in Fig. 356. Similarly for the e. m. f. between the inner and middle slip-rings of Fig. 355 the curve II (Fig. 356) is obtained, and for the e. m. f. between the middle and outer slip-rings of Fig. 355 the curve III (Fig. 356) is obtained. It will be seen that the three resultant curves of e. m. f. I, II, III (Fig. 356) are of equal amplitude, and are displaced relatively to each other by one-third period. The ring winding shown in Fig. 355 is therefore a symmetrical three-phase winding, the winding being obviously a mesh-connection.

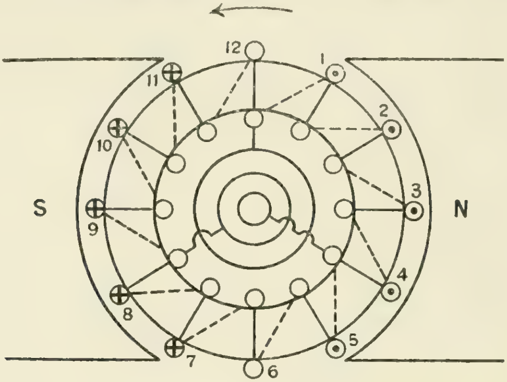


Fig. 355.

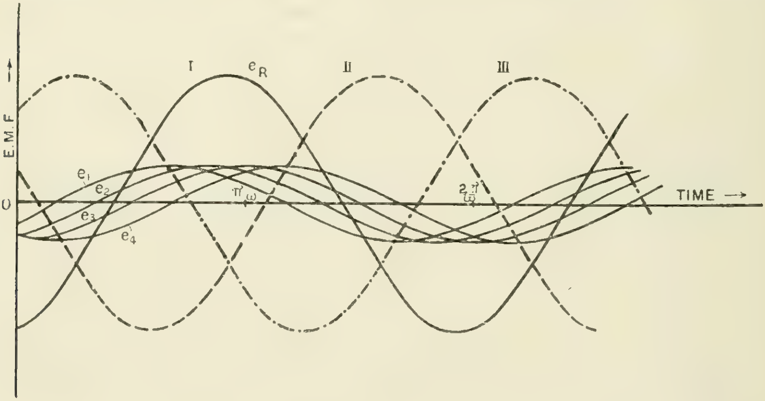


Fig. 356.

The vector diagram for the e. m. f. between the outer and inner slip-rings (Fig. 355) is shown in Fig. 357, where $OE_1 : E_1E_2 :$

E_2E_3 and E_3E_4 are the vectors of e. m. f. induced in the conductors 1, 2, 3, and 4 respectively, the angle between any two consecutive vectors being 30°. The resultant vector is OE_4. Similarly E_4C is the vector of e. m. f. between the inner and middle slip-ring, and CO the vector of e. m. f. between the middle and outer slip-ring.

The winding factor in this case is—

$$\frac{OE_4}{OE_1+E_1E_2+E_2E_3+E_3E_4}=\frac{OE_4}{4\bar{E}},$$

where $\bar{E}$, as before, is the amplitude of the e. m. f. induced in one conductor.

If a circle is drawn through the points OE_4C (Fig. 357) it will be seen that the ends of the vectors OE_1 : E_1E_2 : E_2E_3 : E_3E_4 lies on this circle, and it can be shown that—

$$\frac{OE_4}{4\bar{E}}=0{\cdot}835,$$

which is the winding factor in this particular case.

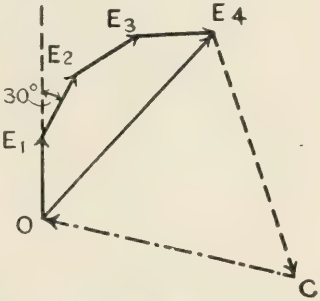


FIG. 357.

If there were an indefinitely large number of conductors in the ring winding of Fig. 355 the vector diagram $OE_1E_2E_3E_4$ would merge into the circle described through the points E_4CO, and the winding factor would be—

$$\frac{\text{chord } OE_4 \text{ of the circle}}{\text{arc } OE_4 \text{ of the circle}}=\frac{3\sqrt{3}}{2\pi}=0{\cdot}825.$$

For a tabulated list of winding factors reference should be made to pp. 407 and 408.

248. Polyphase Ring Winding.—If n equidistant tapping points are made to the armature winding of Fig. 355 (n being greater than 3), each tapping point being taken to a separate slip-ring, such a winding is termed a polyphase mesh-connected winding.

249. Power in Single-Phase, Two-Phase, Three-Phase, and Polyphase Systems.—The instantaneous value of the power in a single-phase system of which the momentary values of p. d. and current are respectively

$$v=\bar{V}\sin\omega t \text{ volts, and } i=\bar{I}\sin(\omega t-\phi) \text{ ampères,}$$

is (§ 233)—

$$w=vi=\bar{V}\bar{I}\sin\omega t\sin(\omega t-\phi)$$

$$=\frac{\bar{V}\bar{I}}{2}[\cos\phi-\cos(2\omega t-\phi)] \text{ watts,}$$

The mean value of this power is—

$$W=\frac{\overline{V}\overline{I}}{2}\cos\phi=VI\cos\phi \text{ watts,}$$

where V and I are root-mean-square values.

And hence $$w=vi=W\left[1-\frac{\cos(2\omega t-\phi)}{\cos\phi}\right] \text{ watts.}$$

The power in a single-phase system is therefore pulsating, as is shown in Fig. 315. The pulsation of the power is, for such purposes as driving motors, a disadvantage of the single-phase system. For other purposes, however, such as lighting, the pulsation is not troublesome if the frequency is not too low, say not less than about 20 cycles per second.

Now consider an n-phase system, and let $v_{\mathrm{I}}:v_{\mathrm{II}}:v_{\mathrm{III}}\ldots v_n$ be the respective phase p. d.'s, $i_{\mathrm{I}}:i_{\mathrm{II}}:i_{\mathrm{III}}\ldots i_n$ the corresponding values of the currents, and $\phi_{\mathrm{I}}:\phi_{\mathrm{II}}:\phi_{\mathrm{III}}\ldots\phi_n$ the phase angles.

The instantaneous value of the total power of the system is—

$$w=v_{\mathrm{I}}i_{\mathrm{I}}+v_{\mathrm{II}}i_{\mathrm{II}}+v_{\mathrm{III}}i_{\mathrm{III}}+\ldots v_n i_n \text{ watts,}$$

and the mean power is—

$$W=V_1I_1\cos\phi_1+V_{\mathrm{II}}I_{\mathrm{II}}\cos\phi_{\mathrm{II}}+\ldots V_nI_n\cos\phi_n \text{ watts.}$$

If each of the n phases is equally loaded, as regards current, power factor, and p. d., the p. d. and current of the xth phase will be—

$$v_x=\overline{V}\sin\left(\omega t-2\pi\frac{x}{n}\right):i_x=\overline{I}\sin\left(\omega t-\phi-2\pi\frac{x}{n}\right)$$

respectively.

$$\text{Hence}\quad w=\overline{V}\overline{I}\sum_{x=1}^{x=n}\left[\sin\left(\omega t-2\pi\frac{x}{n}\right)\sin\left(\omega t-\phi-2\pi\frac{x}{n}\right)\right]$$

$$=\frac{\overline{V}\overline{I}}{2}\left[n\cos\phi-\sum_{x=1}^{x=n}\cos\left(2\omega t-\phi-4\pi\frac{x}{n}\right)\right]$$

$$=\frac{\overline{V}\overline{I}}{2}n\cos\phi \text{ watts,}$$

since $$\sum_{x=1}^{x=n}\cos\left(2\omega t-\phi-4\pi\frac{x}{n}\right)=0.$$

Hence $$w=nVI\cos\phi,$$

where V and I are root-mean-square values.

The total power in a symmetrical balanced n phase load is therefore constant at every instant, and is equal to n times the mean power in one phase.

Example I. Two-Phase Three-Wire System.—Let $v_{\mathrm{I}}=\overline{\mathrm{V}}\sin(\omega t)$ volts, and $v_{\mathrm{II}}=\overline{\mathrm{V}}\sin(\omega t-\frac{\pi}{2})$ volts, see Figs. 331 and 332.

If both phases are equally loaded and have the same power factor—

$$i_{\mathrm{I}}=\overline{\mathrm{I}}\sin(\omega t-\phi)\text{ ampères},$$

$$i_{\mathrm{II}}=\overline{\mathrm{I}}\sin(\omega t-\frac{\pi}{2}-\phi)\text{ ampères},$$

and $w=v_{\mathrm{I}}i_{\mathrm{I}}+v_{\mathrm{II}}i_{\mathrm{II}}$ watts

$$=\overline{\mathrm{V}}\overline{\mathrm{I}}\,[\sin\omega t\sin(\omega t-\phi)+\sin(\omega t-\frac{\pi}{2})\sin(\omega t-\frac{\pi}{2}-\phi)]$$

$$=\frac{\overline{\mathrm{V}}\overline{\mathrm{I}}}{2}[2\cos\phi-\cos(2\omega t-\phi)-\cos(2\omega t-\phi-\pi)]$$

$$=2\mathrm{VI}\cos\phi=2\mathrm{V}_p\mathrm{I}_p\cos\phi\text{ watts},$$

where $\mathrm{V}_p:\mathrm{I}_p$ are r. m. s. values of p. d. and current respectively per phase.

That is to say, the power is constant at *every instant*, and is equal to twice the power in one phase.

The line p. d. is $\overline{\mathrm{V}}_l=\sqrt{2}\overline{\mathrm{V}}_p$—that is, $\mathrm{V}_l=\sqrt{2}\mathrm{V}_p$.

The current in the third wire is $\mathrm{I}_0=\sqrt{2}\,\mathrm{I}_p$ (see p. 352).

The expression for the power may therefore be written—

$$w=\mathrm{W}=2\mathrm{V}_p\mathrm{I}_p\cos\phi=\mathrm{V}_l\mathrm{I}_0\cos\phi\text{ watts},$$

w representing the instantaneous value of the power and W the mean value.

Example II. Three-Phase System.—The power in a three-phase system balanced system—

$$w=\mathrm{W}=3\mathrm{V}_p\mathrm{I}_p\cos\phi\text{ watts}.$$

For mesh connection:

$$\mathrm{V}_p=\mathrm{V}_l:\mathrm{I}_p=\frac{\mathrm{I}_l}{\sqrt{3}}\text{ (see pp. 335 and 337)},$$

and $$w=\mathrm{W}=\sqrt{3}\mathrm{V}_l\mathrm{I}_l\cos\phi\text{ watts}.$$

For star connection:

$$\mathrm{V}_l=\sqrt{3}\mathrm{V}_p:\mathrm{I}_l=\mathrm{I}_p\text{ (see pp. 354 and 355)},$$

and $$w=\mathrm{W}=\sqrt{3}\mathrm{V}_l\mathrm{I}_l\cos\phi\text{ watts}.$$

In the foregoing expressions for the power the quantity considered is the power supplied to the load, as measured by wattmeters. The power developed by the generator, however, will be greater by the amount of power lost in the generator windings.

If in the above expressions for the power the quantities E_p and E_l be respectively inserted instead of V_p and V_l, and ϕ_0 instead of ϕ, the expressions give the total power generated by the alternator.

250. Measurement of the Power supplied to the Load of a Two-Phase System.—A brief description of one common type of wattmeter by means of which the power is measured is given in § 218.

In the case of a two-phase system the power in each phase is measured separately, and the sum of the measurements gives the total power. If the system is balanced the total power is twice the power in one phase (p. 367).

251. Measurement of the Power supplied to the Load of a Three-Phase Balanced System by means of One Wattmeter.—The current coil of the wattmeter is connected in series with one line, say III (Fig. 358), and one end of the pressure coil is connected to the same line. Three non-inductive resistances are then arranged in star connection, as shown in Fig. 358, so that the total resistance in each leg of the star is the same. One leg of the star is connected to the free end of the pressure coil of the wattmeter, and the other two ends are connected respectively to the two mains I and II. The common junction O of the three resistances forms an artificial *neutral point* (§ 240), and the total power supplied to the load will be three times the power measured by the wattmeter.

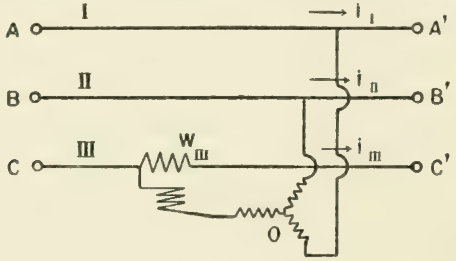


FIG. 358.

If the neutral point of the system is accessible the artificial neutral point is unnecessary, and the wattmeter pressure coil may be connected direct to the neutral point of the system and the line III.

252. Two-Wattmeter Method of measuring Power and Power-Factor in the case of a Three-Phase Balanced Load.—In Fig. 359 the current coil of the wattmeter is shown in series with line I. The pressure coil is connected at one end to line I, and the other end connected alternately to lines II and III by means of the two-way switch *a*.

Let $W_{\mathrm{I\ II}}$ be the wattmeter reading when switch *a* is on 2.
and $W_{\mathrm{I\ III}}$,, ,, ,, ,, ,, ,, 3

The total power supplied to the load is then given by—

$$W_{\mathrm{I\ II}}+W_{\mathrm{I\ III}}.$$

If one of the wattmeter readings, say, $W_{I\ II}$, is negative, the pressure coil terminals should be interchanged to obtain the reading, and the sign of $W_{I\ II}$ taken as negative, the power then being given by the difference—

$$W_{I\ III} - W_{I\ II}.$$

In Fig. 360, let all the vectors represent r. m. s. values, and let $OV_1 : OV_2 : OV_3$ be the vectors of the p. d. of the lines I : II : III respectively with reference to the neutral point O. Let OI_1 be the vector of the current in line I *lagging* by the angle ϕ on the vector OV_1.

From what has been said in § 244 it will be seen that ϕ will also be the phase angle if the load is in mesh connection, since the angle between the phase current and the phase p. d. in star is the same as the angle between the line p. d. and the current in the corresponding phase of the mesh (see Fig. 351, § 244).

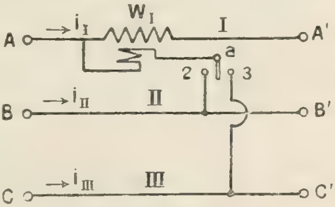


FIG. 359.

The wattmeter readings are therefore—

$$W_{I\ II} = (OV_1V_2)(OI_1)\cos(30^\circ + \phi),$$

and $$W_{I\ III} = (OV_1V_3)(OI_1)\cos(30^\circ - \phi).$$

But $$OV_1V_2 = OV_1V_3 = V_l,$$

and $$I_1 = I_l,$$

hence $$W = W_{I\ III} + W_{I\ II} = V_lI_l\,[\cos(30^\circ - \phi) + \cos(30^\circ + \phi)];$$

or $$W = \sqrt{3}V_lI_l\cos\phi \text{ watts},$$

and thus W is the total power supplied to the load as deduced on p. 367.

The algebraical sum of the two wattmeter readings gives the total power supplied to the load.

Reference to the vector diagram of Fig. 362, which is drawn for a *leading* current, shows that the above result also holds for the case of a balanced three-phase load in which the current leads on the applied p. d.

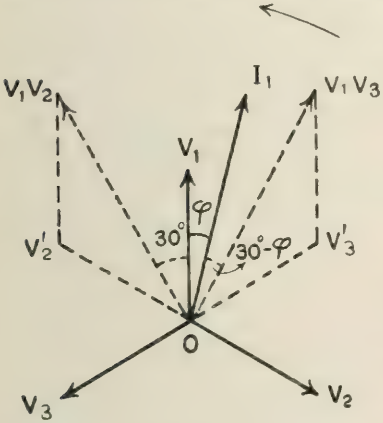


FIG. 360.

If $\phi = 60^\circ$: $W_{I\ II} = 0$ for a lagging current, and $W_{I\ III} = 0$ for a leading current.

If ϕ is greater than 60°, either lagging or leading, one of the wattmeter readings becomes negative. In such a case in practice, when the reading becomes negative in the case of wattmeters that have not a central zero, it would be necessary to reverse the pressure coil connections in order to bring the deflection on to the scale of the instrument, and the reading so obtained would be taken as negative. The total power would then be given by the *difference* of the two wattmeter readings.

The Determination of the Power-Factor from the Readings of the Two Wattmeters—Case I. The Current Lags on the P. D. (Fig. 360)—

Since $W_{\text{I III}} = V_l I_l \cos(30° - \phi)$,

and $W_{\text{I II}} = V_l I_l \cos(30° + \phi)$,

then

$$\begin{aligned} W_{\text{I III}} - W_{\text{I II}} &= V_l I_l [\cos(30° - \phi) - \cos(30° + \phi)] \\ &= 2V_l I_l \sin 30° \sin \phi \\ &= V_l I_l \sin \phi. \end{aligned}$$

Also $W_{\text{I III}} + W_{\text{I II}} = \sqrt{3} V_l I_l \cos \phi$,

hence

$$\tan \phi = \sqrt{3}\, \frac{W_{\text{I III}} - W_{\text{I II}}}{W_{\text{I III}} + W_{\text{I II}}}.$$

This may be written—

$$\tan \phi = \sqrt{3}\, \frac{1 - \dfrac{W_{\text{I II}}}{W_{\text{I III}}}}{1 + \dfrac{W_{\text{I II}}}{W_{\text{I III}}}}.$$

If the ratio $\frac{W_{\text{I II}}}{W_{\text{I III}}}$ be known, the value of $\tan \phi$, and hence the power factor $\cos \phi$, may be deduced from the above formula.

It is convenient to plot the ratio $\frac{W_{\text{I II}}}{W_{\text{I III}}}$ as a function of the corresponding values of $\cos \phi$, as shown in Fig. 361, so that in any given case the power factor may be read off the curve for any given ratio of the two wattmeter readings, it being observed that the reading $W_{\text{I II}}$ is the smaller of the two wattmeter readings for a lagging current, so that the ratio $\frac{W_{\text{I II}}}{W_{\text{I III}}}$ is less than unity throughout the range.

For unity power factor—*i.e.*, $\cos \phi = 1$, $\phi = 0$ in Fig. 360—$W_{\text{I II}} = W_{\text{I III}}$ and the ratio $\frac{W_{\text{I II}}}{W_{\text{I III}}} = 1$.

As ϕ increases for a lagging current the power factor $\cos \phi$ diminishes, $W_{\text{I II}}$ becomes smaller until when $\phi = 60°$ $\cos \phi = 0{\cdot}5$ lagging, $W_{\text{I II}} = 0$. It follows therefore that if one of the wattmeter readings becomes zero, the current is 60° out of phase with the pressure. As ϕ increases still further, the wattmeter reading $W_{\text{I II}}$ becomes negative, and in practice this would mean that the pressure coil connections would have to be reversed in order to get the reading of $W_{\mathrm{I\,II}}$ on the scale of the instrument; the deflection, however, must be taken as of negative sign if such reversal of the pressure coil connections is made. When ϕ becomes 90° lagging (*i.e.*, $\cos\phi=0$), $W_{\mathrm{I\,II}}$ becomes negative and equal in magnitude to $W_{\mathrm{I\,III}}$, and the ratio $\frac{W_{\mathrm{I\,II}}}{W_{\mathrm{I\,III}}}=-1$. The total power is then $=W_{\mathrm{I\,II}}+W_{\mathrm{I\,III}}=0$.

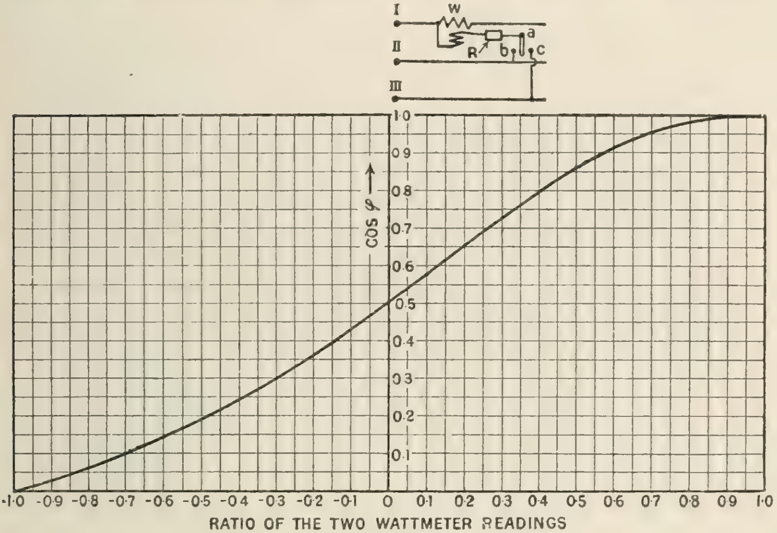


FIG. 361.

Case II. Current Leads on the P. D. (Fig. 362).—If the current leads on the p. d. the wattmeter reading $W_{\mathrm{I\,II}}$ will be greater than the reading $W_{\mathrm{I\,III}}$. When the current leads by 60°, $W_{\mathrm{I\,III}}=0$, and when the current leads by an angle ϕ greater than 60° and less than 90° $W_{\mathrm{I\,III}}$ becomes negative, and the pressure coil connections must be reversed in order to get the deflection for $W_{\mathrm{I\,III}}$ on to the scale of the instrument, such reversal of connections therefore denoting that the corresponding deflection is to be taken with a negative sign.

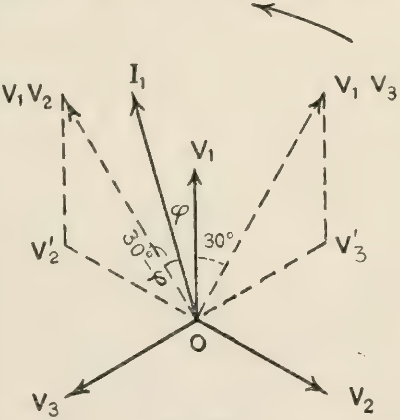


FIG. 362.

The formula for tan ϕ for the case of a leading current may therefore be written—

$$\tan\phi = \sqrt{3}\,\frac{W_{\mathrm{I\,II}} - W_{\mathrm{I\,III}}}{W_{\mathrm{I\,II}} + W_{\mathrm{I\,III}}};$$

that is—

$$\tan\phi = \sqrt{3}\,\frac{1 - \dfrac{W_{\mathrm{I\,III}}}{W_{\mathrm{I\,II}}}}{1 + \dfrac{W_{\mathrm{I\,III}}}{W_{\mathrm{I\,II}}}}.$$

The values of cos ϕ can be read off the curve (Fig. 361) for this case also.

In practice there is nothing to denote directly which are the lines I, II, and III respectively (Fig. 359).

The line in which the current coil of the wattmeter is connected is line I in accordance with the notation used in Figs. 359 to 362, but in order to decide which is line II and which line III a preliminary experiment would be necessary. This could consist of an auxiliary balanced load of choking coils (*i.e.*, coils of large self-induction), and if the current coil of the wattmeter is in line I, when the pressure coil is across line I and II the reading will be less than when the pressure coil is across lines I and III (Fig. 360).

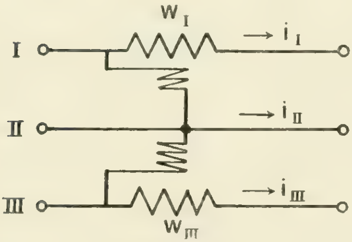


FIG. 363.

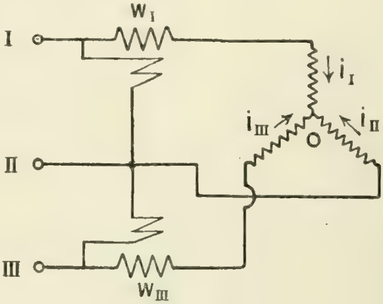


FIG. 364.

In many cases it is known what the general nature of the load is—that is, whether the current may be expected to lag or lead—and in such cases the two wattmeter readings immediately denote which of the two lines is to be numbered II and which III to agree with the assumption of counter-clockwise rotation of the vectors in Figs. 360 and 362.

253. Two-Wattmeter Method of measuring the Power supplied to the Load of a Three-Phase System, whether Balanced or Unbalanced.—For this method the wattmeters are connected in the circuit, as shown in Fig. 363.

It can be proved that the power supplied to the circuit is equal to the algebraical sum of the two wattmeter readings, whether the load is balanced or unbalanced.

Case I. Load is Star-Connected and Unbalanced (Fig. 364).—The power supplied to the circuit at any instant is—

$$w = i_{\text{I}}v_{\text{IO}} + i_{\text{II}}v_{\text{IIO}} + i_{\text{III}}v_{\text{IIIO}} \text{ watts,}$$

where v_{IO} represents the instantaneous value of the pressure of the line I with reference to the star centre O, and similarly for v_{IIO} and v_{IIIO}.

But $$i_{\text{I}} + i_{\text{II}} + i_{\text{III}} = 0;$$

that is— $$i_{\text{II}} = -(i_{\text{I}} + i_{\text{III}}),$$

which means that any one line may be regarded as the return for the currents in the other two lines.

Hence the power at any instant is—

$$\begin{aligned} w &= i_{\text{I}}v_{\text{IO}} - i_{\text{I}}v_{\text{IIO}} - i_{\text{III}}v_{\text{IIO}} + i_{\text{III}}v_{\text{IIIO}} \text{ watts} \\ &= i_{\text{I}}(v_{\text{IO}} - v_{\text{IIO}}) + i_{\text{III}}(v_{\text{IIIO}} - v_{\text{IIO}}) \\ &= i_{\text{I}}v_{\text{I II}} + i_{\text{III}}v_{\text{III II}} \text{ watts.} \end{aligned}$$

But the mean value of $i_{\text{I}}\,v_{\text{I II}}$ is given by the reading of the wattmeter in line I of Fig. 365, and the mean value of $i_{\text{III}}v_{\text{III II}}$ is given by the reading of the wattmeter in line III. Hence the algebraical sum of the two wattmeter readings is a measure of the power supplied to the circuit.

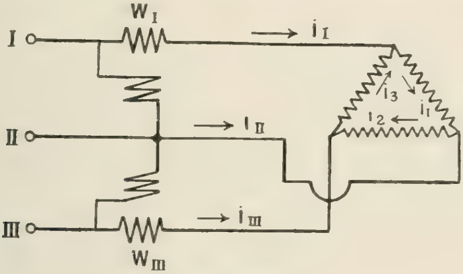


FIG. 365.

If one wattmeter tends to read backwards, its pressure coil leads must be interchanged and the corresponding reading then taken to be negative.

The positive connection of the pressure coil of a wattmeter may easily be found by connecting the wattmeter to a direct current circuit so that the wattmeter reads positive.

Case II. The Load is Mesh-Connected and Unbalanced (Fig. 365).—The power supplied at any instant is—

$$w = v_{\text{I III}}i_3 + v_{\text{III II}}i_2 + v_{\text{II I}}i_1 \text{ watts.}$$

But $$v_{\text{I III}} + v_{\text{III II}} + v_{\text{II I}} = 0. \ (\S\ 149);$$

hence $$v_{\text{I III}} = -(v_{\text{III II}} + v_{\text{II I}}),$$

therefore $w = -v_{\mathrm{III\,II}}i_3 - v_{\mathrm{II\,I}}i_3 + v_{\mathrm{III\,II}}i_2 + v_{\mathrm{II\,I}}i_1$

$= v_{\mathrm{II\,I}}(i_1 - i_3) + v_{\mathrm{III\,II}}(i_2 - i_3)$

$= v_{\mathrm{II\,I}}(i_1 - i_3) + v_{\mathrm{II\,III}}(i_3 - i_2)$ watts.

But $i_{\mathrm{I}} = i_1 - i_3 : i_{\mathrm{III}} = i_3 - i_2$.

Hence $w = v_{\mathrm{II\,I}}i_{\mathrm{I}} + v_{\mathrm{II\,III}}i_{\mathrm{III}}$ watts,

and this is true at every instant.

The sum of the wattmeter reading is therefore a measure of the mean power supplied to the circuit.

In deducing the above results, all the steps hold equally well for a balanced load, and hence the method is applicable to both balanced and unbalanced three-phase circuits.

254. Three-Wattmeter Method of measuring the Power supplied to a Three-Phase Load.—In this method the power in a three-phase unsymmetrical system is measured by means of three wattmeters, as shown in Fig. 366. One wattmeter current coil is connected in series with each line, and the three ends of the pressure coils connected together.

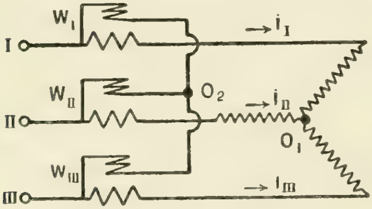


Fig. 366.

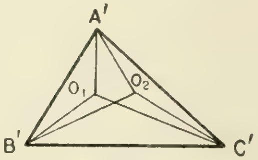


Fig. 367.

In Fig. 367 let $O_1A^1 : O_1B^1 : O_1C^1$ represent the vectors of p. d. between O_1 and the terminals I, II, III respectively (Fig. 366).

Let O_2 correspond to the common junction of the pressure coils.

The sum of the three wattmeter readings is then—

$$w = A^1O_2i_{\mathrm{I}} + B^1O_2i_{\mathrm{II}} + C^1O_2i_{\mathrm{III}}$$
$$= (A^1O_1 + O_1O_2)i_1 + (B^1O_1 + O_1O_2)i_{\mathrm{II}} + (C^1O_1 + O_1O_2)i_{\mathrm{III}}$$
$$= A^1O_1i_1 + B^1O_1i_{\mathrm{II}} + C^1O_1i_3 + (i_1 + i_{\mathrm{II}} + i_{\mathrm{III}})O_1O_2$$
$$= v_{\mathrm{I}}i_{\mathrm{I}} + v_{\mathrm{II}}i_{\mathrm{II}} + v_{\mathrm{III}}i_{\mathrm{III}},$$

since $i_{\mathrm{I}} + i_{\mathrm{II}} + i_{\mathrm{III}} = 0;$

i.e., any one line is the return for the other two.

But $v_{\mathrm{I}}i_{\mathrm{I}} + v_{\mathrm{II}}i_{\mathrm{II}} + v_{\mathrm{III}}i_{\mathrm{III}}$ is the power supplied to the load at any instant.

Hence the sum of the wattmeter readings is a measure of the mean value of total power.

If the neutral point O_1 (Fig. 366) is accessible, the measurement of the total power supplied may be made by means of a single wattmeter. The wattmeter is connected in turn in each line, the pressure coil being connected between the line and the point O_1; the power in each phase is thus measured, and the sum of the readings gives the total power. This method is not always practicable, as it involves disconnecting the circuit at least twice.

It is to be noted that the above methods of measuring power assume that the p. d. and current waves are of sine form. In cases where this assumption cannot reasonably be made the above formula for the power measurement should not be used without further investigation as to the effect of the higher harmonics (see Chapter XIX.) on the wattmeter readings.

255. Transformation of a Two-Phase System to a Three-Phase System—Scott's Method.—It is sometimes required to convert a two-phase supply system to a three-phase supply system, and *vice versa*. The best known method of doing this is the method devised by C. F. Scott.

The method involves the use of two transformers, and for a general consideration of the principle of the transformer reference should be made to Chapter XX., §§ 276 and 277.

In Fig. 368 let AO be the secondary of a transformer T_1, of which the primary winding is *ad*. Let BC be the secondary of a transformer T_2 of which the primary is *bc*. Suppose the end O of the secondary AO be connected to the mid-point of the secondary BC.

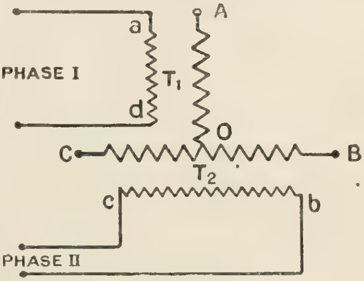


FIG. 368.

Further, let the transformation ratios of the two transformers be such that—

$$\text{for } T_1 \qquad \frac{\text{secondary volts}}{\text{primary volts}} = \frac{\sqrt{3}}{2} = 0{\cdot}866;$$

$$\text{for } T_2 \qquad \frac{\text{secondary volts}}{\text{primary volts}} = 1.$$

Suppose the primaries *ad* and *bc* are connected respectively to the two phases of a symmetrical two-phase supply system.

The pressure of B with respect to C is given by the vector CB, and the pressure of A with respect to O, the mid-point of CB, by the vector OA (Fig. 369).

Since the transformation ratios of the transformers T_1 and T_2 are as $\frac{\sqrt{3}}{2}$: 1, it follows that the vector OA is $\frac{\sqrt{3}}{2}$ times the vector CB or $\sqrt{3}$ times the vector CO, since O is the mid-point of CB.

The pressure of B with respect to C in the secondary of the transformer T_2 is given by the vector CB (Fig. 369). The pressure of A with respect to B is given by the vector OA_B, and is found by subtracting the vector OB from the vector OA (see p. 350).

The pressure of C with respect to A is given by the vector OC_A, and is found by subtracting the pressure of A with respect to O from *the pressure of C with respect to O*—*i.e.*, by reversing both vectors CO and OA and drawing the vector sum, as shown in Fig. 369.

Further, since $OC = OB = \frac{OA}{\sqrt{3}}$, it follows that the vectors OA_B, OC_A, and CB are all equal in magnitude. These vectors are also mutually displaced by one-third of a period (see Fig. 370), and hence the terminals ABC form the terminals of a symmetrical three-phase source of alternating e. m. f.

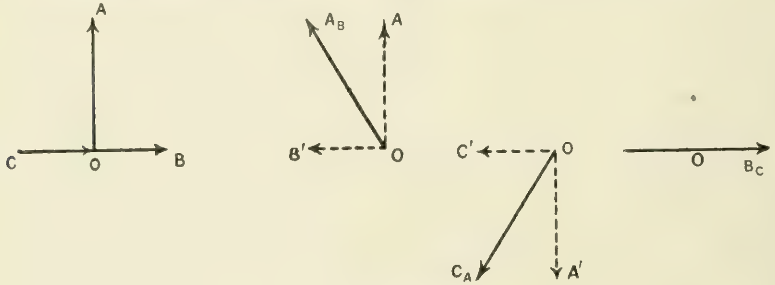


Fig. 369.

The two-phase supply to the primaries of the transformers T_1 and T_2 may be a three-wire system—that is, the ends *c* and *d* may be connected together and to the third wire of the supply.

It is to be observed that if a three-phase balanced load is supplied from the terminals A, B, and C, the load on the two-phase system will also be balanced.

256. The Capacity and Charging Current of Single-Phase Transmission Cables.—When a system of transmission cables is connected to an electric generator, each cable attains the potential of the terminal of the generator to which it is connected. Each cable acquires an electric charge, and an electrostatic field is established in the neighbourhood of each cable. The magnitude of the electric charge depends upon the potential of the cable, and the size and position of the cable with reference to the earth and the neighbouring cables.

When the supply is an alternating the p. d. varies from moment to moment, and the charge on each cable, of course, varies with the potential of the cable.

This varying charge constitutes an electric current which is known as the *charging current* of the cable. The charging current is due to the capacity effects of the cables, and therefore leads on the applied p. d. by one-quarter period.

Reference should be made to §§ 43 and 44, Chapter III., Section I.

In the case of single-phase transmission cables it will be assumed that the cables are relatively distant from the earth, so that the earth has no effect on the electrostatic field of the cables.

Let A and B (Fig. 371) be the two cables distant a apart, the diameter of each cable being d.

If at any instant the cable A has a charge $+q$ units per cm. length, the cable B will have a charge $-q$ units per cm. length. There will therefore be an electrostatic field between the cables, as illustrated in Fig. 45.

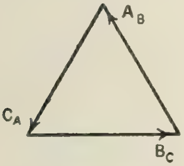


FIG. 370.

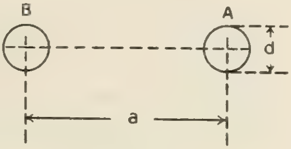


FIG. 371.

If v is the p. d. between the cables at any instant and q the charge per cm. length, both quantities being measured in electrostatic units (see §§ 28 and 36), the ratio $\frac{q}{v}$ is the capacity C of the electrostatic field per cm. length of cable. For a given size and disposition of the cables the ratio $\frac{q}{v}$ is a constant—

i.e., $$\frac{q}{v} = \text{C electrostatic units.}$$

If q is reduced to coulombs and v to volts, C will be given in farads.

Thus q electrostatic units $= \frac{q}{3 \times 10^9}$ coulombs (§ 36);

and v electrostatic units $= 300v$ volts (§ 28);

hence C in electrostatic units $= \frac{1}{9 \times 10^{11}}$ C farads

$$= \frac{1}{9 \times 10^5} \text{ C micro-farads.}$$

The value of C for the two cables in Fig. 371 was found in p. 48 to be—

$$C = \frac{1}{4 \log_e \left[\frac{\sqrt{a^2 - d^2} + a}{d} \right]} \text{ electrostatic units per cm. length of cable}$$

if the cables are in air—*i.e.*, $\epsilon = 1$.

Usually a is large compared with d, and then very approximately

$$C = \frac{1}{4 \log_e \left(\frac{2a}{d}\right)} \text{ electrostatic units per cm. length of cable,}$$

or $$C = \frac{10^5}{9 \times 10^5 \times 4 \times 2{\cdot}3 \log_{10} \left(\frac{2a}{d}\right)} \text{ micro-farads per kilometre;}$$

that is, $C = \frac{0{\cdot}012 l}{\log_{10} \left(\frac{2a}{d}\right)}$ micro-farads if l is length in kilometres,

or $$C = \frac{0{\cdot}0194 l}{\log_{10} \left(\frac{2a}{d}\right)} \text{ micro-farads if } l \text{ is length in miles,}$$

as previously given on p. 45.

If the alternating p. d. between the cable is given by—

$$v = \overline{\overline{V}} \sin \omega t \text{ volts,}$$

the charging current will be—

$$i = \omega \frac{C}{10^6} \overline{\overline{V}} \cos \omega t \text{ ampères (p. 328),}$$

if C is in micro-farads;

or $$\overline{\overline{I}} = \frac{\omega C \overline{\overline{V}}}{10^6} \text{ ampères,}$$

which is the maximum value of the sine wave of the charging current.

Example.—Let $l = 5$ miles: $\overline{\overline{V}} = 10000$ volts: $f = \frac{\omega}{2\pi} = 50$ cycles per second: $a = 26$ inches: $d = \frac{1}{2}$ inch.

$$C = \frac{0{\cdot}097}{\log_{10} 104} \text{ micro-farads}$$

$$= 0{\cdot}047 \text{ micro-farad,}$$

and $$\overline{\overline{I}} = \frac{314 \times 0{\cdot}047 \times 10000}{10^6} \text{ ampères}$$

$$= 0{\cdot}147 \text{ ampère.}$$

257. The Capacity and Charging Current of Three-Phase Transmission Cables.

Case I. It is assumed that the cables are relatively so far removed from the earth that the influence of the earth on the capacity of the electrostatic field of the cables is negligible. It is also assumed that the distance between the cables is large compared with the diameter of each cable.

In Fig. 372 the three cables are represented by the circles I, II, and III respectively, the axes of the cables being denoted by O_1, O_2, and O_3 respectively. The point O is symmetrically placed with regard to the three circles I, II, and III.

If at any instant t the charge on cable I is $Q \sin \omega t$ per cm. length, the charges on cables II and III will be $Q \sin\left(\omega t - \frac{2\pi}{3}\right)$, and $Q \sin\left(\omega t - \frac{4\pi}{3}\right)$ respectively.

If a charge $+1$ be considered as placed on the line OO_1 and moved from the point O to the surface A_1 of cable I, the work done in ergs in making the transfer gives the potential, in electrostatic units, of cable I with respect to O.

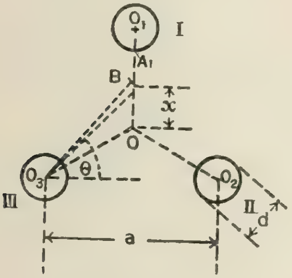


FIG. 372.

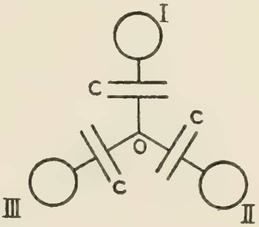


FIG. 373.

Then if $q = Q \sin \omega t$ electrostatic units is the charge per cm. length on cable I, and v = work done in ergs in moving the charge $+1$ electrostatic unit from O to A_1, the ratio $\frac{q}{v}$ will be the capacity, in electrostatic units, of the field per cm. length of cable between cable I and the line through O parallel to the cable.

The capacity of the system of cables may thus be represented by the three condensers C (Fig. 373) star-connected between the cables and the point O.

The magnitude of the capacity C may be found by considering the electric forces on a charge $+1$ electrostatic unit placed at any point B on the line OO_1 (Fig. 372).

These forces are—

(i.) $\dfrac{2Q}{O_1B}\sin\omega t$ dynes due to the charge on cable I (see § 25), and acting in the direction O_1O.

(ii.) $\dfrac{2Q}{O_2B}\sin\left(\omega t-\dfrac{2\pi}{3}\right)$ dynes due to the charge on cable II and acting in the direction O_2B.

(iii.) $\dfrac{2Q}{O_3B}\sin\left(\omega t-\dfrac{4\pi}{3}\right)$ dynes due to the charge on cable III and acting in the direction O_3B.

The resultant force on the charge +1 unit at B is therefore—

$$F=\left[\frac{2Q}{O_1B}\sin\ \omega t-\frac{2Q}{O_2B}\sin\left(\omega t-\frac{2\pi}{3}\right)\sin\theta-\frac{2Q}{O_3B}\right.$$
$$\left.\sin\left(\omega t-\frac{4\pi}{3}\right)\sin\theta\right]\text{ dynes,}$$

where θ is the angle shown in Fig. 372,

and $$\sin\theta=\frac{d(O_2B)}{dx}=\frac{d(O_3B)}{dx}.$$

The work done in moving the charge +1 from O to A_1 is therefore—

$$\int_{O}^{OA_1}Fdx=\left[2Q\sin\omega t\int_{O}^{OA_1}\frac{dx}{O_1B}-2Q\sin\left(\omega t-\frac{2\pi}{3}\right)\int_{O_2O}^{O_2A_1}\frac{d(O_2B)}{O_2B}\right.$$
$$\left.-2Q\sin\left(\omega t-\frac{4\pi}{3}\right)\int_{O_3O}^{O_3A_1}\frac{d(O_3B)}{O_3B}\right]\text{ ergs.}$$

This reduces to—

$$\int_{O}^{OA_1}Fdx=2Q\sin\omega t\log_e\frac{O_2A_1}{O_1A_1}\text{ ergs,}$$

since $2Q\sin\omega t+2Q\sin\left(\omega t-\dfrac{2\pi}{3}\right)+2Q\sin\left(\omega t-\dfrac{4\pi}{3}\right)=0.$

The capacity C (Fig. 373) is therefore given by—

$$C=\frac{q}{v}=\frac{Q\sin\omega t}{2Q\sin\omega t\log_e\dfrac{O_2A_1}{O_1A_1}}$$

in electrostatic units per cm. of cable.

If a is large compared with d (Fig. 373) so that O_2A_1 is very approximately equal to a, the equivalent star capacity C per phase is—

$$C = \frac{0{\cdot}039\epsilon}{\log_{10}\left(\frac{2a}{d}\right)} \text{ micro-farads per mile of cable,}$$

where ϵ is the dielectric constant.

It thus appears that the equivalent star capacity C (Fig. 373) of a three-phase cable is *twice* the capacity between the two lines of a single-phase transmission system of which the diameter of each line is d and the distance between the two lines is a (see p. 378).

Now it can be proved by similar reasoning to that employed in the determination of the value of C (Fig. 373) that the capacity between *any two lines* of the three-phase cable (Fig. 372) is *two-thirds* the capacity between the lines of a single-phase system of which the diameter of each line is d and the distance between the lines is a.

It follows, therefore, that the equivalent star capacity C (Fig. 373) of a three-phase cable is *three times* the capacity between any two lines (see also p. 470).

Case II. Three-Phase Cable provided with an Earthed Sheath.—In this case it is necessary to consider the capacity between each cable and the sheath as well as between the cables.

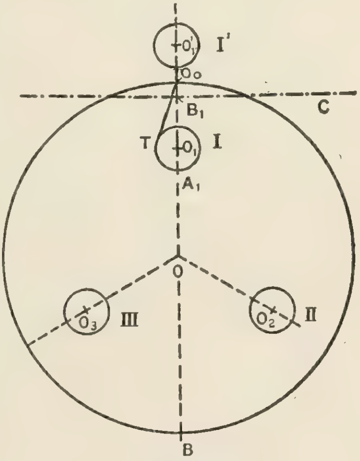


FIG. 374.

In Fig. 374 the three circle centres O_1, O_2, O_3 represent the three cables I, II, III, and the large circle the sheath. In order to find the capacity of, say, cable I to the sheath, the method of inversion explained in § 43 is used. That is to say, the *image* of cable I in the sheath is obtained. To do this the circles O_1 and O_0B are inverted with reference to O_0 so that the circle O_1 inverts into itself and the circle O_0B into the straight line B_1C. The condition to be fulfilled so that the two required inversions may be obtained is—

$$(O_0T)^2 = O_0B_1 \times O_0B,$$

as shown on pp. 50 and 51.

The electric field between cable I and the earthed sheath and due to a charge on I, is the same as the field between the cable I and the earthed plane B_1C. But the field between cable I and this plane is the same as the field between cable I and the image of I in B_1C—viz., I^1 where I and I^1 are equidistant from the plane B_1C (see p. 53 and Fig. 56)—

i.e., $$B_1O_1{}^1 = B_1O_1.$$

If this procedure is followed for the three cables I, II, III, the equivalent arrangement shown in Fig. 375 is obtained.

The equivalent capacity between cable I and the line through the symmetrical point O may then be found by considering the work done in moving an electric charge +1 from O to A_1.

If the cable I has a charge $Q \sin \omega t$, the image I^1 will have a charge $-Q \sin \omega t$. The cables II and II^1 will have charges $Q \sin\left(\omega t - \frac{2\pi}{3}\right)$ and $-Q \sin\left(\omega t - \frac{2\pi}{3}\right)$ respectively, and the cables III and III^1 the charges $Q \sin\left(\omega t - \frac{4\pi}{3}\right)$ and $-Q \sin\left(\omega t - \frac{4\pi}{3}\right)$ respectively.

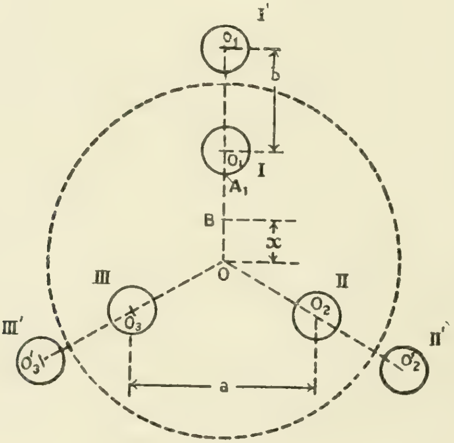


FIG. 375.

The forces in dynes on a charge +1 at B are therefore

$$+\frac{2Q}{O_1B} \sin \omega t \text{ due to I}: \; -\frac{2Q}{O_1{}^1B} \sin \omega t \text{ due to } I^1$$

$$+\frac{2Q}{O_2B} \sin\left(\omega t - \frac{2\pi}{3}\right) \text{ in direction } O_2B \text{ due to II}$$

$$-\frac{2Q}{O_2{}^1B} \sin\left(\omega t - \frac{2\pi}{3}\right) \text{ in direction } O_2{}^1B \text{ due to II}$$

$$+\frac{2Q}{O_3B} \sin\left(\omega t - \frac{4\pi}{3}\right) \text{ in direction } O_3B \text{ due to III}$$

$$-\frac{2Q}{O_3{}^1B} \sin\left(\omega t - \frac{4\pi}{3}\right) \text{ in direction } O_3{}^1B \text{ due to } III^1.$$

If F dynes be the resultant force in direction O_1O, the work done in moving the charge $+1$ from O to A_1 will be—

$$\int_{O}^{OA_1} F\,dx \text{ ergs},$$

and this will be the potential of I with respect to O.

If this integral be determined, as in the previous case, the equivalent capacity of cable I to O is found to be—

$$C = \frac{1}{2\left[\log_e \frac{O_3A_1}{O_1A_1} - \log_e \frac{O_3^1A_1}{O_1^1A_1}\right]}$$

in electrostatic units per cm. length of cable;

or $$C = \frac{0{\cdot}039l\epsilon}{\log_{10} \frac{O_3A_1}{O_1A_1} - \log_{10} \frac{O_3^1A_1}{O_1^1A_1}} \text{ micro-farads,}$$

where l is the length of the cable in *miles*, and ϵ is the specific inductive capacity of the insulating material.

This result is approximate in so far as it has been assumed that the section of each core is small compared with the distance between any two cores. The assumption has also been made that the surfaces of the conductors are smooth cylindrical surfaces.

Example.—Let $l=10$ miles: $\epsilon=4$: diameter of cable sheath $=1$ inch : diameter of each core $=\frac{1}{8}$ inch : distance between centres of any two cores $=0{\cdot}41$ inch : $O_3A_1=0{\cdot}383$ inch : $O_1A_1=0{\cdot}0625$ inch : $O_3^1A_1=0{\cdot}735$ inch : $O_1^1A_1=0{\cdot}439$ inch.

The equivalent star capacity of this cable will be—

$$C=2{\cdot}76 \text{ micro-farads per phase.}$$

Let the supply pressure per phase be $v_p=\bar{V}_p \sin \omega t$

where $$\bar{V}_p=2{,}000 \text{ volts};\ \omega=2\pi f=314.$$

The amplitude of the charging current per phase will be—

$$\bar{I}_p=\omega \frac{C}{10^6} \bar{V}_p \text{ for C in micro-farads,}$$

that is $$\bar{I}_p=\frac{314\times 2{\cdot}76\times 2{,}000}{10^6}=1{\cdot}73 \text{ ampères per phase,}$$

and this is the current which will flow in the cable on open circuit.

CHAPTER XIX

HARMONIC ANALYSIS

258. Non-sinusoidal Wave Forms.—Although, as was stated in § 224, the sine wave forms of current and e. m. f. are the standards in electrical engineering, it is seldom that these ideal wave forms are actually obtained in practice, although close approximation are frequently obtained. The wave forms met with in practice are frequently widely different from the sine shape, and it is necessary to consider how to deal with such non-sinusoidal waves.

259. Harmonic Analysis.—By Fourier's theorem it is known that any continuous single-valued periodic function, whatever its form, may be represented by a series of sine curves of different frequency, phase, and amplitude.

If $f(t)$ is any periodic, continuous, single-valued function of the time t, then by Fourier's theorem—

$$f(t) \equiv \mathrm{K} + \mathrm{K}_1 \sin(\omega t + \theta_1) + \mathrm{K}_2 (\sin 2\omega t + \theta_2) + \ldots$$

where the sign $\equiv$ means "identically equal to," $\mathrm{K}, \mathrm{K}_1, \mathrm{K}_2 \ldots$ are constants, and $\omega = 2\pi \times$ frequency (see p. 323).

If the axis of t divides the curve of a single complete period into a positive and negative half of equal area, the constant K is zero, and this is the condition commonly met with in alternating current and alternating e. m. f. waves.

The term $\mathrm{K}_1 \sin(\omega t + \theta_1)$ is the component sine wave of lowest frequency, and is called the *fundamental* wave. The remaining terms are called *harmonics* of the first term from the analogy to the "overtones" in acoustics.

The second term $\mathrm{K}_2 \sin(2\omega t + \theta_2)$ is of double frequency, and, strictly speaking, is the first harmonic; the third term of triple frequency is the second harmonic, and so on. It is usually the practice, however, to call the second term of *double frequency the second harmonic*, and so on. The terms of frequencies which are multiples of two are called the even harmonics, and the other terms the odd harmonics.

The periodic function—

$$f(t) \equiv \mathrm{K}_1 (\sin \omega t + \theta_1) + \mathrm{K}_2 \sin(2\omega t + \theta_2) + \mathrm{K}_3 \sin(3\omega t + \theta_3) + \ldots$$

may be written—

$$f(t) \equiv \mathrm{A}_1 \sin \omega t + \mathrm{A}_2 \sin 2\omega t + \mathrm{A}_3 \sin 3\omega t + \ldots$$

$$+ \mathrm{B}_1 \cos \omega t + \mathrm{B}_2 \cos 2\omega t + \mathrm{B}_3 \cos 3\omega t + \ldots$$

where $K_1=\sqrt{A_1^2+B_1^2} : K_2=\sqrt{A_2^2+B_2^2} : K_3=\sqrt{A_3^2+B_3^2} \ldots$
and $K_1 \cos \theta_1=A_1 : K_2 \cos \theta_2=A_2 : K_3 \cos \theta_3=A_3 \ldots$
$K_1 \sin \theta_1=B_1 : K_2 \sin \theta_2=B_2 : K_3 \sin \theta_3=B_3 \ldots$

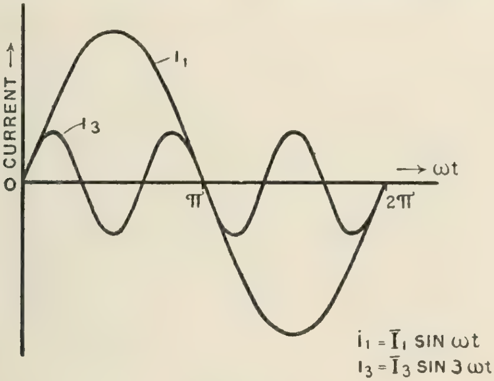
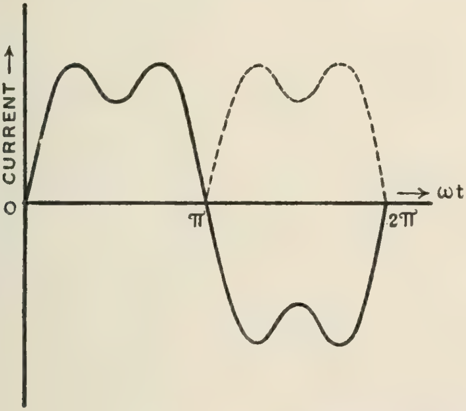


FIG. 376.

By writing the series in this form it is seen that all the sine terms pass through zero and all the cosine terms pass through their maxima values simultaneously. The amplitudes of the different harmonics may be found by the analytical method explained in § 261.



$i = \bar{I}_1 \sin \omega t + \bar{I}_3 \sin 3\omega t$

FIG. 377.

260. Characteristic Effects of Odd and Even Harmonics respectively on the shape of the Resultant Wave.—In Fig. 376 is shown a fundamental sine wave of current i_1 and a triple frequency wave of current i_3. The resultant of these two waves is shown in Fig. 377.

It will be observed that if the positive half of the wave of Fig. 377 be displaced along the abscissa axis by a half period so as to occupy the position shown dotted, this displaced half wave becomes an exact image of the negative half wave in the abscissa axis. In other words, the value of the current in Fig. 377 for the angle ωt is equal and opposite to the value of the current for the angle $(\omega t + \pi)$. This is characteristic of all periodic waves containing only *odd harmonics*, and is the type of wave which is most frequently met with in electrical engineering problems. Another example of such a wave is shown in Fig. 378. It is thus possible to predict at once that on analysing such a wave as that of Fig. 378 into its component waves, only odd harmonics will be found.

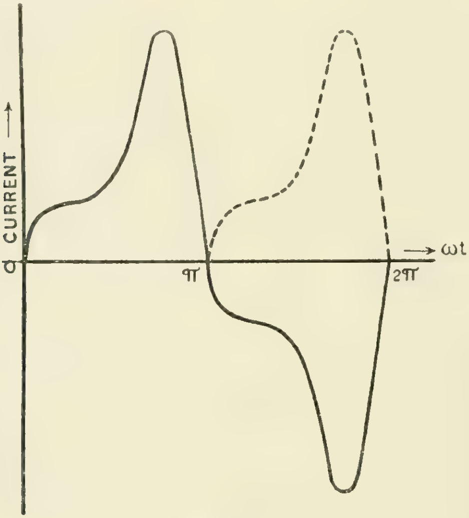


FIG. 378.

In Fig. 379 is shown a fundamental sine wave of current i_1 and a double frequency current wave i_2.

The resultant wave is shown in Fig. 380, and it is to be observed that if the positive half of the wave in Fig. 380 be displaced along the abscissa axis by half a period so as to occupy the position shown dotted, the displaced half wave is *not* an image in the abscissa axis of the negative half wave.

The value of the current, however, in Fig. 380 for the angle ωt is equal and opposite to the value of the current for the angle $2\pi - \omega t$. This is characteristic of all periodic wave forms containing even harmonics; such wave forms, however, are comparatively seldom met with in electrical engineering practice.

An example of such a wave is shown in Fig. 391, which is the resultant of the two waves—

$$i_1 = 30 \sin 4\omega t : i_2 = 20 \sin 5\omega t.$$

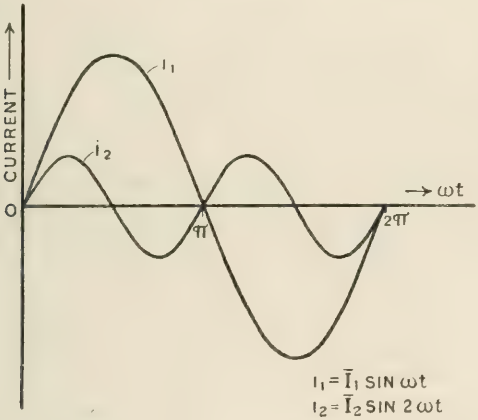


Fig. 379.

The frequency of these waves is respectively—

$$f_1 = \frac{4\omega}{2\pi} \text{ and } f_2 = \frac{5\omega}{2\pi}.$$

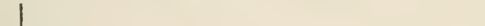


Fig. 380.

An inspection of Figs. 390, 391, and 392 will show that the frequency of the resultant wave is—

$$f = \frac{5\omega - 4\omega}{2\pi} = f_2 - f_1.$$

This method of obtaining a low-frequency wave from two superposed waves of slightly different frequencies is known as the "beat" effect, and is a well-known phenomenon in acoustics.

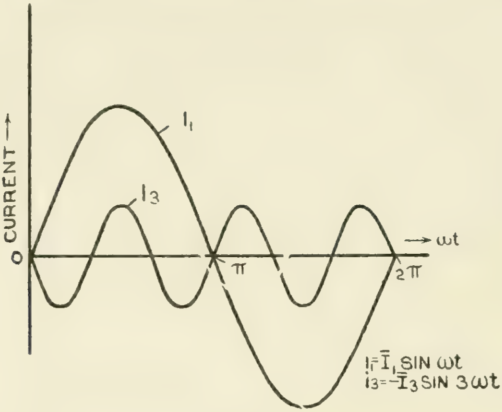


Fig. 381.

In Fig. 381 are shown the waves—

$$i_1=\bar{I}_1 \sin \omega t : i_3 = -\bar{I}_3 \sin 3\omega t,$$

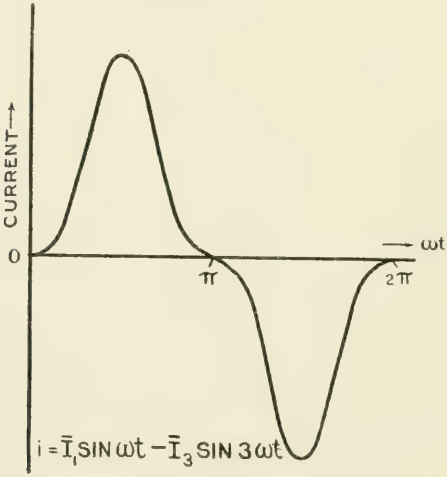


Fig. 382.

and in Fig. 382 the resultant of these two superposed waves.

In Fig. 383 are shown the current waves—

$$i_1=\bar{I}_1 \sin \omega t : i_5 = I_5 \sin 5\omega t.$$

In Fig. 384 the resultant of the two current waves of Fig. 383 is shown.

Fig. 385 shows the current waves—

$$i_1 = \bar{I}_1 \sin \omega t : i_5 = -\bar{I}_5 \sin 5\omega t,$$

and Fig. 386 shows the resultant of these two waves.

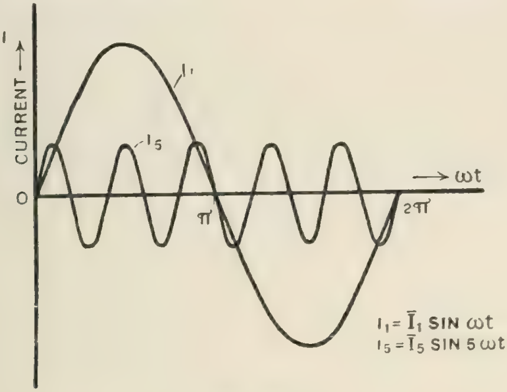


Fig. 383.

261. Analytical Method of resolving any given Continuous, Single-Valued, Periodic Curve into its Component Sine Waves.—Numerous methods have been devised for analysing periodic curves into the component sine waves. The method explained in the following will be found to be simple and convenient. As has already been stated in § 260, it is possible by inspection

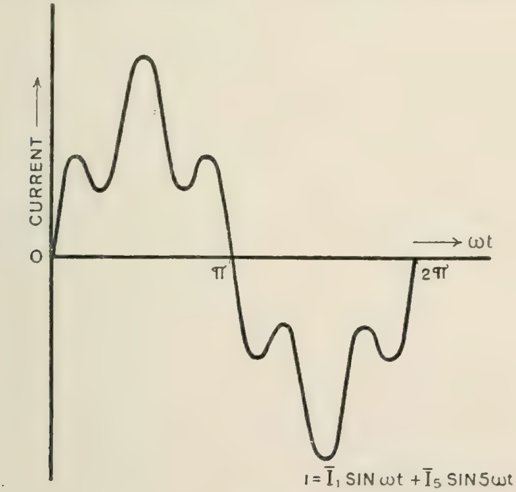


Fig. 384.

to predict whether there are any even harmonics in the curve, and the two cases will be considered separately—viz., (1) when only odd harmonics are present, and (2) when even and odd harmonics are present. From what follows it will be clear that Case II. holds generally whether even harmonics are present or not, but since the arithmetical work involved in Case I. is considerably less than in Case II., the process of analysis may be appreciably shortened if, by a preliminary inspection of the curve, it is ascertained that only odd harmonics are to be expected.

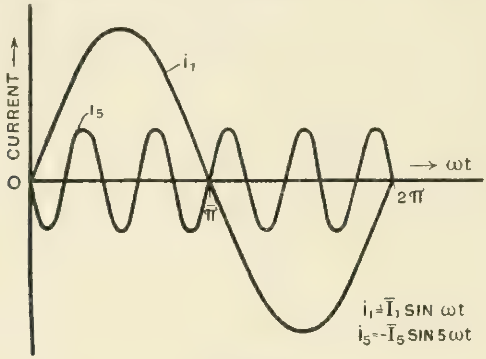


FIG. 385.

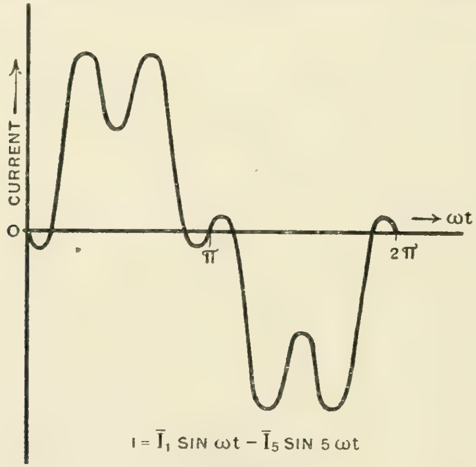


FIG. 386.

Case I. Only Odd Harmonics Present.—In the first place consider the integral—

$$\int_0^\pi \sin mx \sin nx \, dx,$$

where *both m and n are odd* whole numbers.

This may be written—

$$\tfrac{1}{2}\int_0^\pi [\cos (m-n)x - \cos (m+n)x]\, dx.$$

If m is not equal to n the value of this expression is zero if m and n are *both odd* or *both even*, since each of the cosine terms goes through a complete cycle of positive and negative values a whole number of times in the range over which the integration is taken, and hence the integration of each term is zero.

If, however, m is equal to n, the first term becomes unity, and the integration of the second term zero.

Hence

$$\int_0^\pi \sin^2 nx\, dx = \tfrac{1}{2}\int_0^\pi dx = \frac{\pi}{2}.$$

Similarly it may be shown that—

$$\int_0^\pi \sin mx \cos nx\, dx = 0$$

for all values of m and n which are *both odd* or *both even* whole numbers (see also Case II., p. 394).

Further $$\int_0^\pi \cos^2 nx\, dx = \frac{\pi}{2},$$

and $$\int_0^\pi \cos mx \cos nx\, dx = 0,$$

for *all* values of m and n which are whole numbers but are not equal.

Now consider the curve $f(x)$ of which the analysis is required and which has been found by inspection to contain only *odd* harmonics—that is,

$$\begin{aligned} f(x) \equiv {} & A_1 \sin x + A_3 \sin 3x + \ldots \\ & + B_1 \cos x + B_3 \cos 3x + \ldots \end{aligned}$$

Suppose the series be multiplied by $\sin nx$ and integrated with respect to x from $x=0$ to $x=\pi$.

That is— $$\int_0^\pi f(x) \sin nx\, dx \equiv$$

$$\int_0^\pi \begin{bmatrix} A_1 \sin x \sin nx + \ldots + A_n \sin^2 nx + \ldots \\ + B_1 \cos x \sin nx + \ldots + B_n \cos nx \sin nx + \ldots \end{bmatrix} dx.$$

From what has been proved in the foregoing it will be seen that the integration of all the terms on the right-hand side is zero with the exception of

$$\int_0^\pi A_n \sin^2 nx\, dx,$$

and this term is equal to—

$$A_n \frac{\pi}{2}.$$

Hence $$A_n = \frac{2}{\pi}\int_0^\pi f(x) \sin nx\, dx.$$

Similarly if the series be multiplied by cos nx—

$$B_n = \frac{2}{\pi}\int_0^\pi f(x) \cos nx\, dx,$$

and this gives the method of determining each coefficient of the series of sine and cosine terms which is equivalent to the given periodic function of x.

The process then is as follows:

Take m equally spaced ordinates in the half cycle of the curve, the distance between the ordinates being therefore $\frac{\pi}{m}$.

Let $y_1, y_2, y_3 \;\ldots$ be the values of the respective ordinates. Then—

$$A_n = \frac{2}{\pi}\frac{\pi}{m}\left[y_1 \sin n\frac{\pi}{m} + y_2 \sin n\frac{2\pi}{m} + \ldots + y_{m-1} \sin \frac{n(m-1)\pi}{m} + y_m \sin n\frac{m\pi}{m}\right];$$

that is, since in the last term of the above expression, $\sin n\pi = 0$,

$$A_n = \frac{2}{m}\left[y_1 \sin n\frac{\pi}{m} + y_2 \sin n\frac{2\pi}{m} + \ldots\; y_{m-1} \sin \frac{n(m-1)\pi}{m}\right]$$

and

$$B_n = \frac{2}{m}\left[y_1 \cos n\frac{\pi}{m} + y_2 \cos n\frac{2\pi}{m} + \ldots - y_m\right].$$

Example 1.—Suppose it is required to analyse the current curve given in Fig. 387. It will be seen by inspection (see § 260) that there are *only odd* harmonics in this curve. The values of the abscissæ of the curve are given in seconds and also in degrees, so that 360° corresponds to one cycle—viz., $\frac{2\pi}{\omega}$ seconds. Take nine equispaced ordinates—*i.e.*, $m = 9$: $\frac{\pi}{m} = 20°$.

Write down a table of values as follows:

Ordinates.	*Fundamental Wave.*		*Third Harmonic.*	
$i_1=19$	i_1 sin 20° = 6·5	i_1 cos 20° = 17·9	i_1 sin 60° = 16·4	i_1 cos 60° = 9·5
$i_2=31$	i_2 sin 40° = 20·6	i_2 cos 40° = 23·9	i_2 sin 120° = 26·9	i_2 cos 120° = − 15·5
$i_3=38$	i_3 sin 60° = 33	i_3 cos 60° = 19·0	i_3 sin 180° = 0	i_3 cos 180° = − 38
$i_4=44$	i_4 sin 80° = 43·2	i_4 cos 80° = 7·6	i_4 sin 240° = − 38·1	i_4 cos 240° = − 22
$i_5=49$	i_5 sin 100° = 48·2	i_5 cos 100° = − 8·5	i_5 sin 300° = − 42·5	i_5 cos 300° = 24·5
$i_6=53$	i_6 sin 120° = 45·8	i_6 cos 120° = − 26·5	i_6 sin 360° = 0	i_6 cos 360° = 53
$i_7=55{\cdot}5$	i_7 sin 140° = 37·0	i_7 cos 140° = − 42·5	i_7 sin 420° = 48	i_7 cos 420° = 27·7
$i_8=31$	i_8 sin 160° = 10·6	i_8 cos 160° = − 29	i_8 sin 480° = 26·9	i_8 cos 480° = − 15·5
$i_9=0$	i_9 sin 180° = 0	i_9 cos 180° = 0	i_9 sin 540° = 0	i_9 cos 540° = 0
	Sum = 244·9	Sum = − 38·1	Sum = 37·6	Sum = 23·7

Hence
$$A_1=\frac{244{\cdot}9}{4{\cdot}5}=54{\cdot}2:\ A_3=\frac{37{\cdot}6}{4{\cdot}5}=8{\cdot}25,$$

$$B_1=-\frac{38{\cdot}1}{4{\cdot}5}=-8{\cdot}5:\ B_3=\frac{23{\cdot}7}{4{\cdot}5}=5{\cdot}3.$$

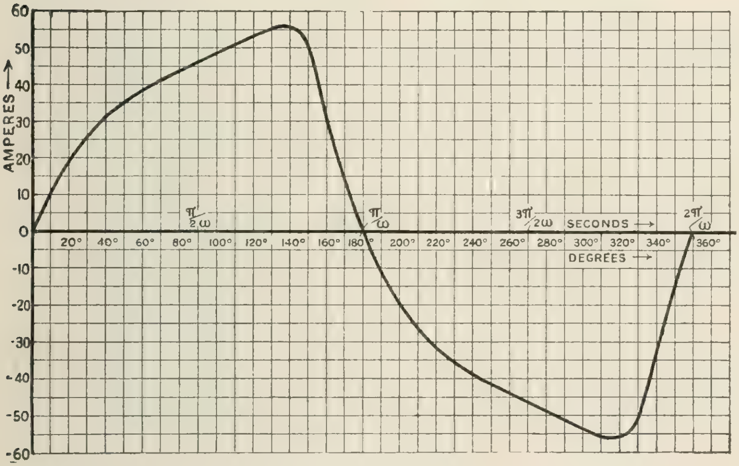


Fig. 387.

A similar analysis yields for the fifth harmonic—

$$A_5=0{\cdot}5:\ B_5=3{\cdot}05,$$

and similarly for the coefficients of the higher harmonics.

The curve of Fig. 387 may therefore be represented by the series—

$$54{\cdot}2\sin\omega t+8{\cdot}25\sin 3\omega t+0{\cdot}5\sin 5\omega t$$
$$-8{\cdot}5\cos\omega t+5{\cdot}3\cos 3\omega t+3{\cdot}05\cos 5\omega t.$$

A still closer approximation may be obtained by taking the harmonics higher than the fifth into account.

The above series may be rewritten—

$$54{\cdot}9 \sin (\omega t - 8^\circ\ 50') + 9{\cdot}82 \sin (3\omega t + 32^\circ\ 45')$$
$$+ 3{\cdot}1 \sin (5\omega t + 80^\circ\ 40'),$$

and this series is shown in Fig. 388.

By adding corresponding ordinates of the waves in Fig. 388 and comparing with the ordinates of the original curve of Fig. 387, it will be seen what degree of approximation is made by neglecting harmonics higher than the fifth.

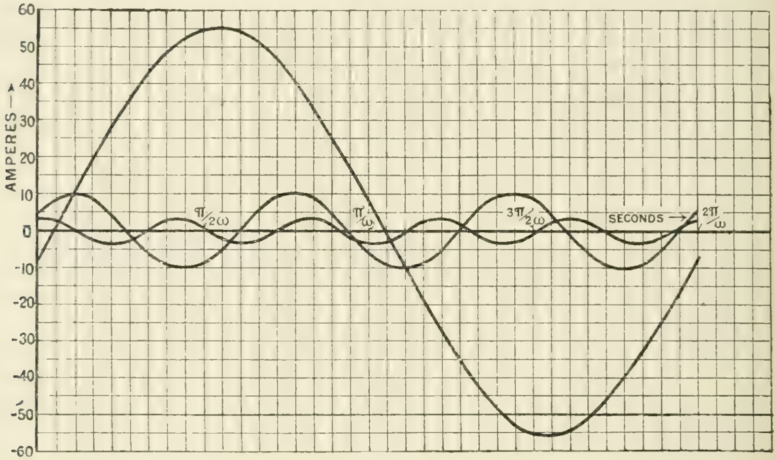


Fig. 388.

Example 2.—In Fig. 389 is shown a current wave which, by analysis as far as the fifth harmonic, is found to be equivalent to the following series:

$$29{\cdot}5 \sin \omega t + 11{\cdot}1 \sin 3\omega t - 0{\cdot}54 \sin 5\omega t$$
$$- 12{\cdot}6 \cos \omega t + 8{\cdot}5 \cos 3\omega t + 3{\cdot}85 \cos 5\omega t,$$

which may be written—

$$32{\cdot}1 \sin (\omega t - 23^\circ) + 14 \sin (3\omega t + 37\tfrac{1}{4}^\circ) - 3{\cdot}87 \sin (5\omega t - 82^\circ).$$

These component waves are shown in Fig. 390.

Case II. When Even Harmonics are Present.—If *either m or n is an even* whole number, the value of the integral—

$$\int_0^\pi \sin mx \cos nx \, dx$$

is not zero.

If, however, the integration limits be chosen as $-\pi$ to $+\pi$, the value of the integral is—

$$\int_{-\pi}^{+\pi} \sin mx \cos nx \, dx = 0,$$

for all values of m and n which are whole numbers, whether *odd or even.*

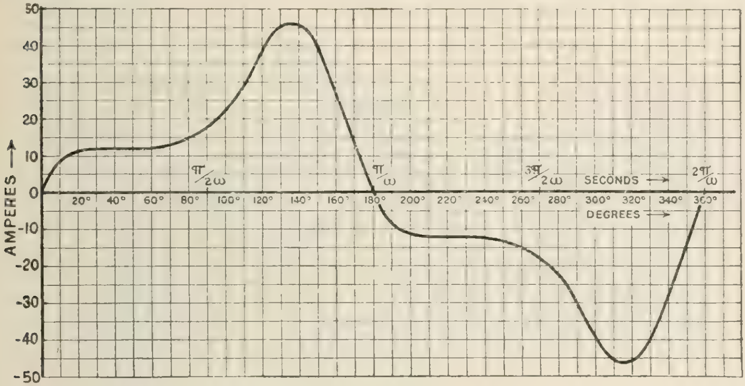


Fig. 389.

Further, $\int_{-\pi}^{+\pi} \sin mx \sin nx \, dx = 0$ for all values of m and n which are not equal. If $m = n$, but not equal to zero,

$$\int_{-\pi}^{+\pi} \sin mx \sin nx \, dx = \int_{-\pi}^{+\pi} \sin^2 nx \, dx = \pi.$$

Also $\int_{-\pi}^{+\pi} \cos mx \cos nx \, dx = 0$ for all values of m and n which are not equal. If $m = n$ but not equal to zero,

$$\int_{-\pi}^{+\pi} \cos mx \cos nx \, dx = \int_{-\pi}^{+\pi} \cos^2 nx \, dx = \pi.$$

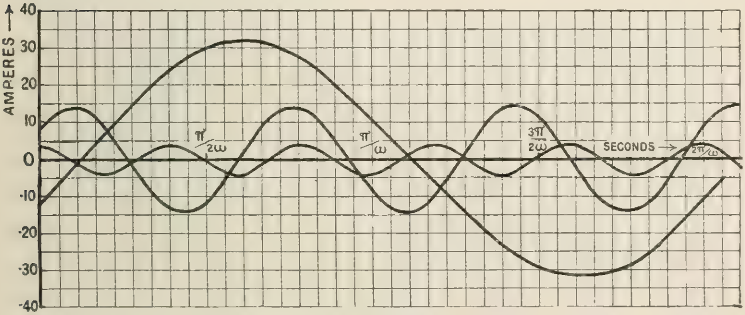


Fig. 390.

Hence, to find the values of the coefficients A and B in this case, the series $f(x)$ is multiplied by $\sin nx$ to obtain the coefficient A_n, and by cos nx to obtain the coefficient B_n and the integration of the product taken between the limits $-\pi$ and $+\pi$.

$$\text{Thus}\quad A_n=\frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin nx\,dx : B_n=\frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx\,dx.$$

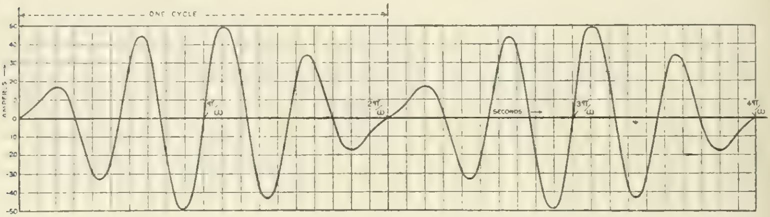


Fig. 391.

Example.—In Fig. 391 is shown a periodic curve which is at once seen to contain one or more even harmonics, and on analysis is found to be represented by the component terms

$$30 \sin 4\omega t - 20 \sin 5\omega t,$$

which are shown in Fig. 392.

This is an example of the superposition of two waves of slightly different frequency which result in a wave of frequency equal to the difference of frequencies of the two component

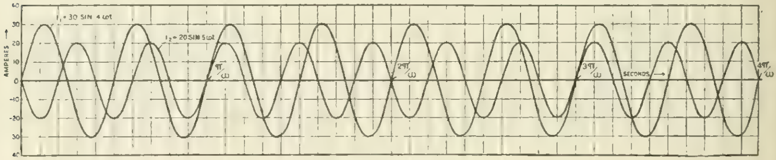


Fig. 392.

waves. The maximum value of the resultant wave is about equal to the sum of the maxima of the two components. This effect of obtaining a relatively low-frequency wave by superposing two high-frequency waves of slightly different frequencies is known as the beat effect, and is a well-known phenomenon in acoustics, as when two tuning-forks of slightly different pitch are sounded together. The frequency of the resultant wave is the difference of the frequencies of the two component waves (see p. 387).

For further examples of analysed periodic waves, see Chapter XX, §§ 262 and 264.

CHAPTER XX

MAGNETIC FIELDS DUE TO ALTERNATING CURRENTS

262. Magnetic Field due to a Concentrated, Single-Phase, Alternating Current Winding.—In Fig. 393 let the outer circle represent the periphery of an iron core provided with two slots SS_1 in which a coil is wound. The inner circle represents the periphery of an iron core concentric with the outer core, a uniform air-gap thus being included between the two cores.

In what follows it will be assumed that the permeability of the iron is very great, so that the whole magnetomotive force is effective in driving the flux across the air-gap (see p. 267).

Suppose the coil has w turns and let i ampères be the current in the coil at any moment. The magnetomotive force round any such path as is shown dotted in Fig. 393 will be—

$$\frac{4\pi}{10}\,iw,$$

as explained in § 192.

If δ cms. is the length of the air gap, and B the flux density in the gap,

then
$$\frac{4\pi}{10}\,iw=2\delta B,$$

or
$$B=\frac{2\pi iw}{10\delta},$$

and this value of the flux density is the same at any instant throughout the gap. This is shown by the rectangular diagram A*acgh*A in Fig. 394, where ABCD represents the developed periphery of the inner core of Fig. 393.

If
$$i=\bar{I}\sin\omega t,$$

the height of the rectangle in Fig. 394 is at any instant t, is—

$$B_t=\frac{2\pi w}{10\delta}\,\bar{I}\sin\omega t.$$

If the rectangular figure A*acgh*A be analysed into its component sine waves by the method given in Chapter XIX., it will be found that it can be represented at any instant by the series of sine waves—

$$\frac{4}{\pi}\,B_t\,[\sin\theta+\tfrac{1}{3}\sin 3\theta+\tfrac{1}{5}\sin 5\theta+\;.\;.\;.\;.$$

This series of sine waves up to the fifth harmonic is shown in Fig. 394.

As B_t varies with the time the height of the rectangle increases and diminishes correspondingly, but the rectangle can always be represented by the above series of sine curves for any height B_t.

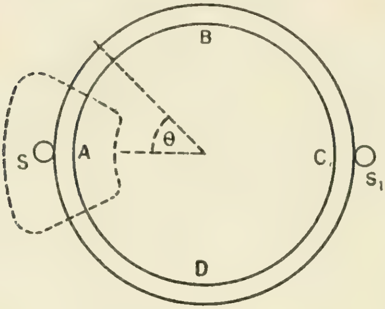


FIG. 393.

In practice the winding is so arranged that the curve of flux distribution is much more nearly a sine wave than the rectangle *AacghA*—that is, the higher harmonics are much less prominent. In the following discussion the higher harmonics will, in the first place, be neglected, and the field assumed to be represented solely by the fundamental sine wave. Brief reference will then be made to the effect of the higher harmonics.

As the current $i=\bar{I}\sin wt$ in the winding SS_1 (Fig. 393) pulsates the flux density in the air-gap pulsates proportionately.

Let F_t be the amplitude of the fundamental wave of flux distribution *at any instant t*, so that F_t is the height BE in Fig. 394.

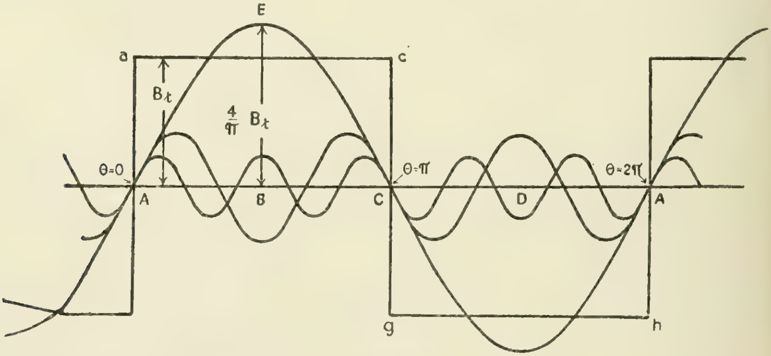


FIG. 394.

Then $$F_t=\bar{F}\sin\omega t,$$

where $\bar{F}=\frac{4}{\pi}\times$ (the *maximum* value of B_t during one cycle of the current).

That is, $$\bar{F}=\frac{4}{\pi}\frac{2\pi w}{10\delta}\bar{I}=0\cdot 8\,\frac{w}{\delta}\bar{I};$$

and $\overline{\mathrm{F}}$ is accordingly the maximum value which F_t attains during one cycle of the current supply.

As a first approximation, therefore—*i.e.*, neglecting the higher harmonics shown in Fig. 394—it is seen that the flux density in the air-gap of Fig. 393 due to any current in the coil SS_1 is distributed along the periphery ABCDA, according to a sine law, so that the maximum value of the flux density is at the points B and D midway between A and C.

When the current in the coil SS_1 varies as a sine function of the time the flux distribution along the periphery ABCDA remains sinusoidal, and the amplitude of this sine wave varies as a sine function of the time.

Reference to Fig. 395 (*a–e*) will make this clearer. In each of these figures the developed periphery of the air-gap of Fig. 393 is marked ABCDA. The wave marked F denotes the sine wave of flux distribution in the air-gap of Fig. 393. The current supplied to the coil SS_1 (Fig. 393) is assumed to vary as a sine function of the time—viz.:

$$i=\overline{\mathrm{I}} \sin \omega t,$$

and in Fig. 395 (*a–e*) the wave F is shown for the times

$$t=0 : \frac{\pi}{4\omega} : \frac{\pi}{2\omega} : \frac{3\pi}{4\omega} : \text{and } \frac{\pi}{\omega} \text{ seconds respectively.}$$

In accordance with the previous notation the amplitude of the wave F is therefore as follows:

Fig. 395*a*, $t=0$; amplitude of $\mathrm{F}=\overline{\mathrm{F}} \sin 0=0$.

Fig. 395*b*, $t=\frac{\pi}{4\omega}$; amplitude of $\mathrm{F}=\overline{\mathrm{F}} \sin \frac{\pi}{4}=\frac{\overline{\mathrm{F}}}{\sqrt{2}}$.

Fig. 395*c*, $t=\frac{\pi}{2\omega}$; amplitude of $\mathrm{F}=\overline{\mathrm{F}} \sin \frac{\pi}{2}=\overline{\mathrm{F}}$.

Fig. 395*d*, $t=\frac{3\pi}{4\omega}$; amplitude of $\mathrm{F}=\overline{\mathrm{F}} \sin \frac{3\pi}{4}=\frac{\overline{\mathrm{F}}}{\sqrt{2}}$.

Fig. 395*e*, $t=\pi$; amplitude of $\mathrm{F}=\overline{\mathrm{F}} \sin \pi=0$.

263. Resolution of an Alternating Field into Two Equal and Oppositely Rotating Fields.—In Fig. 395 each diagram of the left-hand row represents the same arrangement of cores and winding as Fig. 393, the winding now being denoted by the circles A and C. Each diagram of the right-hand row shows the development ABCDA of the periphery of the air-gap.

Now consider two fields F_1 and F_2 sinusoidally distributed along the periphery ABCDA, each of constant amplitude $\frac{\overline{\mathrm{F}}}{2}$ and rotating in opposite directions with an angular velocity ω as shown in the two rows of diagrams of Fig. 395.

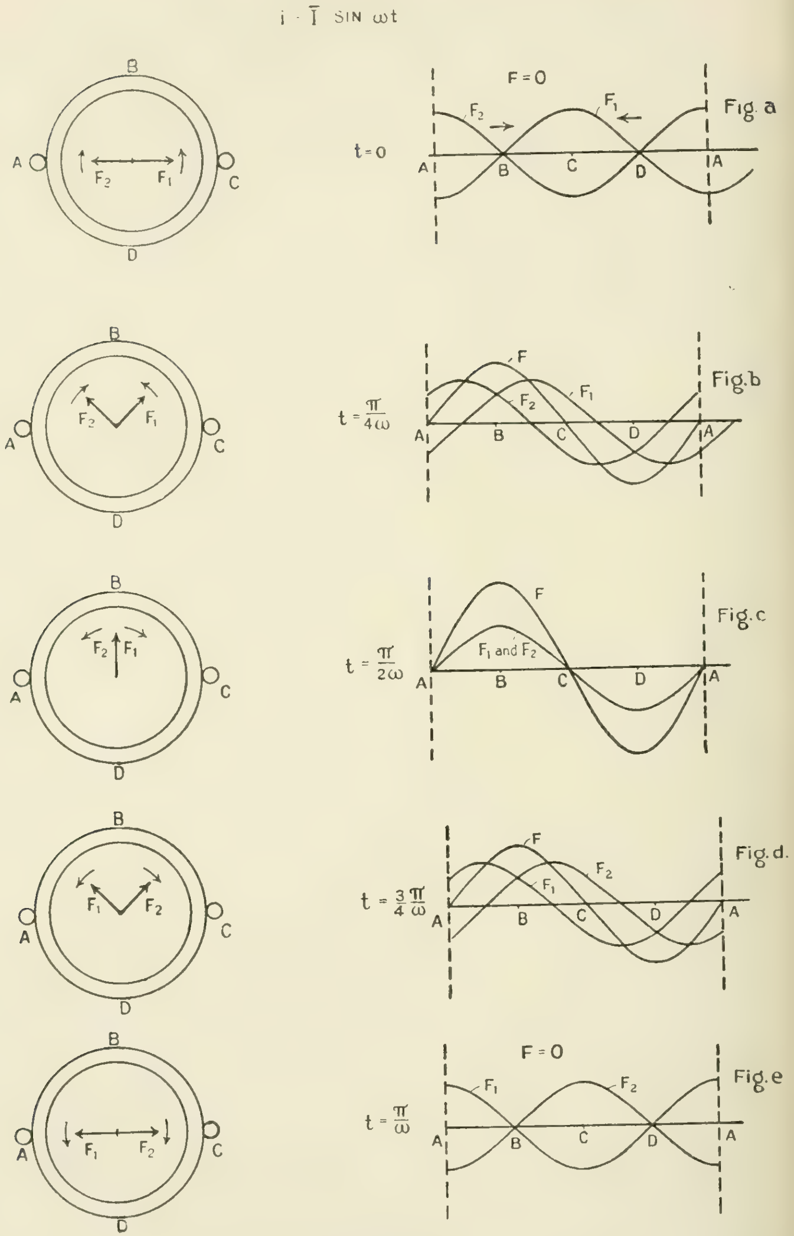


FIG. 395.

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Suppose at time $t=0$, F_1 and F_2 have the relative positions shown in the top diagrams of each row. Since at this instant they are oppositely directed, the resultant field is zero. At the time $t=\frac{\pi}{4\omega}$ the relative positions of the two fields will be as shown in the second diagram of each row, and the resultant field F has therefore an amplitude equal to twice the amplitude of each of the rotating fields F_1 and F_2 multiplied by $\frac{1}{\sqrt{2}}$. That is, for time $t=\frac{\pi}{4\omega}$ the amplitude of the resultant field is—

$$F=2\frac{\overline{F}}{2}\frac{1}{\sqrt{2}}=\frac{\overline{F}}{\sqrt{2}}.$$

Similarly for the times $t=\frac{\pi}{2\omega}$, $\frac{3\pi}{4\omega}$, and $\frac{\pi}{\omega}$ second respectively.

It thus appears that the two equal and oppositely rotating fields F_1 and F_2 are equivalent to the pulsating field F. The result may be stated as follows:

An alternating field of which the space distribution of flux density in a cylindrical gap is sinusoidal of maximum amplitude $\overline{F}$, *is equivalent to two equal fields uniformly rotating in opposite directions, each of sinusoidal space distribution and each of constant amplitude* $\frac{\overline{F}}{2}$.

The resolution of an alternating field into two equal and oppositely rotating fields is very useful when considering the performance of a single-phase motor.

264. Magnetic Field due to a Distributed, Single-Phase, Alternating Current Winding.—Suppose the winding has w turns in series and wound in two pairs of holes HL and KM (Fig. 396), such that the angle between HL and KM is θ, there being $\frac{w}{2}$ turns concentrated in each pair of holes.

The two fundamental sine waves of flux distribution due to the $\frac{w}{2}$ turns in each pair of holes will be relatively displaced by the angle θ (Fig. 397), and consequently at any instant the resultant wave will have an amplitude (*i.e.*, a peak or crest value) less than twice the amplitude of the individual waves—*i.e.*, less than the amplitude of the wave which would be produced if the w turns were concentrated in one pair of holes, such as SS_1 (Fig. 393).

Let the amplitude of the individual waves (Fig. 397) at any instant be F^1. The amplitude of the resultant wave will be—

$$F=2F^1\cos\frac{\theta}{2}.$$

The ratio $\frac{F}{2F^1}$, which in this case is $\cos\frac{\theta}{2}$, is the *winding factor* for the fundamental wave, and will be denoted by f_{w1} in what follows.

The winding factor is, therefore, the ratio of the amplitude of the resultant sinusoidal wave of flux distribution at any instant to the amplitude which the wave would have at that instant if the whole of the w turns of the winding were concentrated in a single pair of holes.

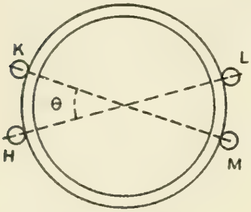


FIG. 396.

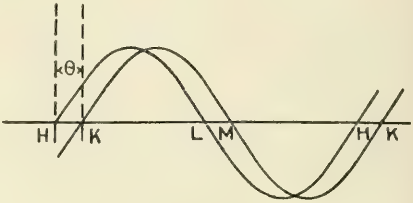


FIG. 397.

If the winding is distributed evenly over the whole of the periphery of the core, the curve of distribution of flux density in the air-gap will be the triangular-shaped diagram A*b*C*d*A (Fig. 398).

If the current in the winding at any instant is i ampères and the length of the air-gap is δ cms., the height B*b* of the triangle at that instant will be—

$$B_t = \frac{2\pi i w}{10\delta}.$$

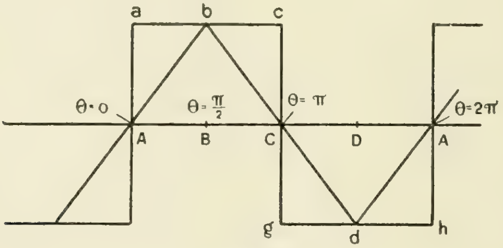


FIG. 398.

Reference to p. 398 (Fig. 394) will show that this is the same as the height of the rectangle A*ac*C for the case in which the winding is wholly concentrated in one pair of holes, as in Fig. 393.

The triangular wave A*b*C*d*A (Fig. 398), which is the curve of space distribution of flux density at any instant, may be resolved in a series of sine curves by the use of the method explained in Chapter XIX. The series so obtained is as follows:

$$\frac{8}{\pi^2}\,\mathrm{B}_t\left[\sin\theta-\frac{1}{9}\sin 3\theta+\frac{1}{25}\sin 5\theta-\frac{1}{49}\sin 7\theta+\ .\ .\ .\right],$$

where B_t is the height of Bb (Fig. 398) at the instant considered.

The ratio of the amplitude of the fundamental wave of this series to the amplitude of the fundamental for a concentrated winding is the winding factor f_{w1} for the fundamental wave.

It was stated in § 262 (Fig. 394) that for a concentrated winding the amplitude of the fundamental wave of space distribution of flux density in the air-gap is—

$$\frac{4}{\pi}\,\mathrm{B}_t.$$

Hence $$f_{w1}=\frac{\frac{8}{\pi^2}}{\frac{4}{\pi}}=\frac{2}{\pi}=0{\cdot}636.$$

It will be observed that for the distributed winding the higher harmonics are relatively much less prominent than is the case for the concentrated winding. Thus, the amplitude of the third harmonic is only $\frac{1}{9}$ the amplitude of the fundamental, whereas with a concentrated winding the third harmonic has an amplitude $\frac{1}{3}$ that of the fundamental (see § 262). The other harmonics are still less prominent with the distributed winding.

From the two series of sine curves into which the rectangular wave of flux distribution (Fig. 394), and the triangular wave (Fig. 398) may be respectively analysed, the winding factors of the higher harmonics may be written down at once.

Thus, for the third harmonic $f_{w3}=\dfrac{\frac{4}{\pi}\,\frac{1}{3}}{\frac{8}{\pi^2}\,\frac{1}{9}}=-0{\cdot}212,$

,, ,, fifth ,, $f_{w5}=\dfrac{\frac{4}{\pi}\,\frac{1}{5}}{\frac{8}{\pi^2}\,\frac{1}{25}}=0{\cdot}127,$

,, ,, seventh ,, $f_{w7}=\dfrac{\frac{4}{\pi}\,\frac{1}{7}}{\frac{8}{\pi^2}\,\frac{1}{49}}=-0{\cdot}091,$

and for the ninth harmonic $f_{w9}=\dfrac{\frac{4}{\pi}\,\frac{1}{9}}{\frac{8}{\pi^2}\,\frac{1}{81}}=0{\cdot}07.$

265. Winding Factors for Concentrated and Distributed, Single-Phase, Alternating Current Windings.—In Fig. 399 suppose the windings to be concentrated in three pairs of holes spaced equally at angles θ. The fundamental waves of the flux distribution curves will be relatively displaced by the angle θ, and the amplitude of the resultant fundamental wave will be

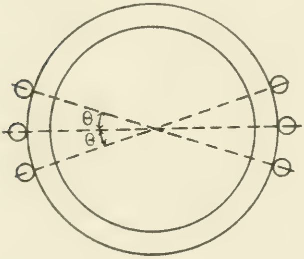


Fig. 399.

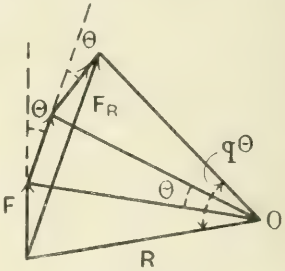


Fig. 400.

the vector sum of the three fundamentals as shown by F_R in Fig. 400, where F is the amplitude of the fundamental wave of flux distribution due to the winding in one pair of diametrically opposite holes (Fig. 399).

The winding factor is therefore—

$$\frac{F_R}{3F}=\frac{2R\sin\left(3\dfrac{\theta}{2}\right)}{3\times 2R\sin\dfrac{\theta}{2}}=\frac{\sin 3\dfrac{\theta}{2}}{3\sin\dfrac{\theta}{2}}.$$

If the winding is concentrated in q pairs of holes—*i.e.*, q holes per pole equally spaced by the angle θ—the winding factor will be—

$$f_{w1}=\frac{\sin q\dfrac{\theta}{2}}{q\sin\dfrac{\theta}{2}}.$$

If there are Q holes per pole of which q consecutive holes are wound, the spacing angle will be $\dfrac{\pi}{Q}$ electrical radians and the winding factor will be—

$$f_{w1}=\frac{\sin\dfrac{q}{Q}\dfrac{\pi}{2}}{q\sin\dfrac{1}{Q}\dfrac{\pi}{2}}.$$

For the third harmonic of the flux wave it will be seen by reference to Fig. 401 that if the fundamental waves are displaced by an angle θ electrical degrees, the third harmonics will be displaced by an angle 3θ electrical degrees. The winding factor for the third harmonic is therefore—

$$f_{w3}=\frac{\sin q\, 3\,\frac{\theta}{2}}{q \sin 3\,\frac{\theta}{2}};$$

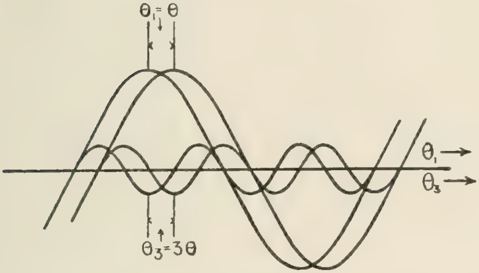


Fig. 401.

or if there are Q equally spaced holes per pole of which only q consecutive holes are wound, then

$$f_{w1}=\frac{\sin \frac{q}{Q}\frac{\pi}{2}}{q \sin \frac{1}{Q}\frac{\pi}{2}},$$

$$f_{w3}=\frac{\sin 3\,\frac{q}{Q}\frac{\pi}{2}}{q \sin \frac{3}{Q}\frac{\pi}{2}},$$

$$f_{w5}=\frac{\sin 5\,\frac{q}{Q}\frac{\pi}{2}}{q \sin \frac{5}{Q}\frac{\pi}{2}},$$

$$f_{w7}=\frac{\sin 7\,\frac{q}{Q}\frac{\pi}{2}}{q \sin \frac{7}{Q}\frac{\pi}{2}},$$

and so on.

If, instead of the winding being concentrated in holes, it is uniformly distributed over the periphery of the core, as is very closely the case in armatures wound for direct current, then if

τ represents the pole pitch and S the breadth of winding per pole, the angle subtended by the winding per pole at the centre of the core will be—

$$a = \frac{S}{\tau}\pi.$$

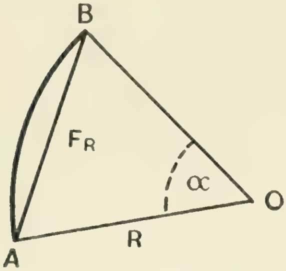


Fig. 402.

In this case the vector diagram becomes an arc of a circle (Fig. 402), and the winding factor for the fundamental wave is the ratio—

$$f_{w1} = \frac{\text{chord AB}}{\text{arc AB}} = \frac{2R\sin\frac{a}{2}}{Ra} = \frac{\sin\frac{a}{2}}{\frac{a}{2}}$$

or

$$f_{w1} = \frac{\sin\frac{S}{\tau}\frac{\pi}{2}}{\frac{S}{\tau}\frac{\pi}{2}}.$$

Similarly for the harmonics—

$$f_{w3} = \frac{\sin 3\frac{S}{\tau}\frac{\pi}{2}}{3\frac{S}{\tau}\frac{\pi}{2}}$$

$$f_{w5} = \frac{\sin 5\frac{S}{\tau}\frac{\pi}{2}}{5\frac{S}{\tau}\frac{\pi}{2}},$$

and so on.

The tables on pp. 407 and 408 have been prepared from the foregoing formulæ.

In Table I. the angular displacement between the holes is $\frac{\pi}{Q}$ electrical radians, and the winding is concentrated in q consecutive holes per pole.

In Table II. the winding is uniformly distributed. The angle subtended at the centre of the core by the breadth of the winding per pole is $\frac{S}{\tau}\pi$.

266. Magnetic Field due to Three-Phase, Alternating Current Windings.—In Fig. 403 let the three-phase winding be represented by the coils $11^1 : 22^1 : 33^1$, each phase being concentrated in one pair of holes, and the coils being mutually displaced by $\frac{2}{3}\pi$ radians. The development of the winding is

TABLE I

Winding Concentrated in Holes

Number of holes per pole Q . .	1	2	3	3	4	4	4	6	6	6	8	8	8	8
Winding concentrated in q holes per pole where q is	1	2	2	3	2	3	4	2	4	6	2	4	6	8
Winding factor f_{w_1}	1	0·707	0·866	0·666	0·925	0·804	0·654	0·966	0·835	0·644	0·985	0·906	0·794	0·641
Winding factor f_{w_3}	1	−0·353	0·000	−0·333	0·385	−0·118	0·270	0·707	0·000	−0·236	0·833	0·319	−0·115	−0·450
Winding factor f_{w_5}	1	−0·353	−0·866	0·666	−0·385	−0·136	−0·270	0·259	−0·224	0·172	0·556	−0·212	−0·077	0·150

TABLE II

DISTRIBUTED WINDING

$\frac{S}{\tau}$	0·2	0·4	0·6	0·8	1·0
f_1	0·986	0·937	0·857	0·756	0·636
fw_3	0·860	0·504	0·109	– 0·156	– 0·212
fw_5	0·636	0·000	– 0·222	0·000	0·127

shown in Fig. 403 at the top of the right-hand row of diagrams. The positive direction of the current in each phase is as shown by the arrows.

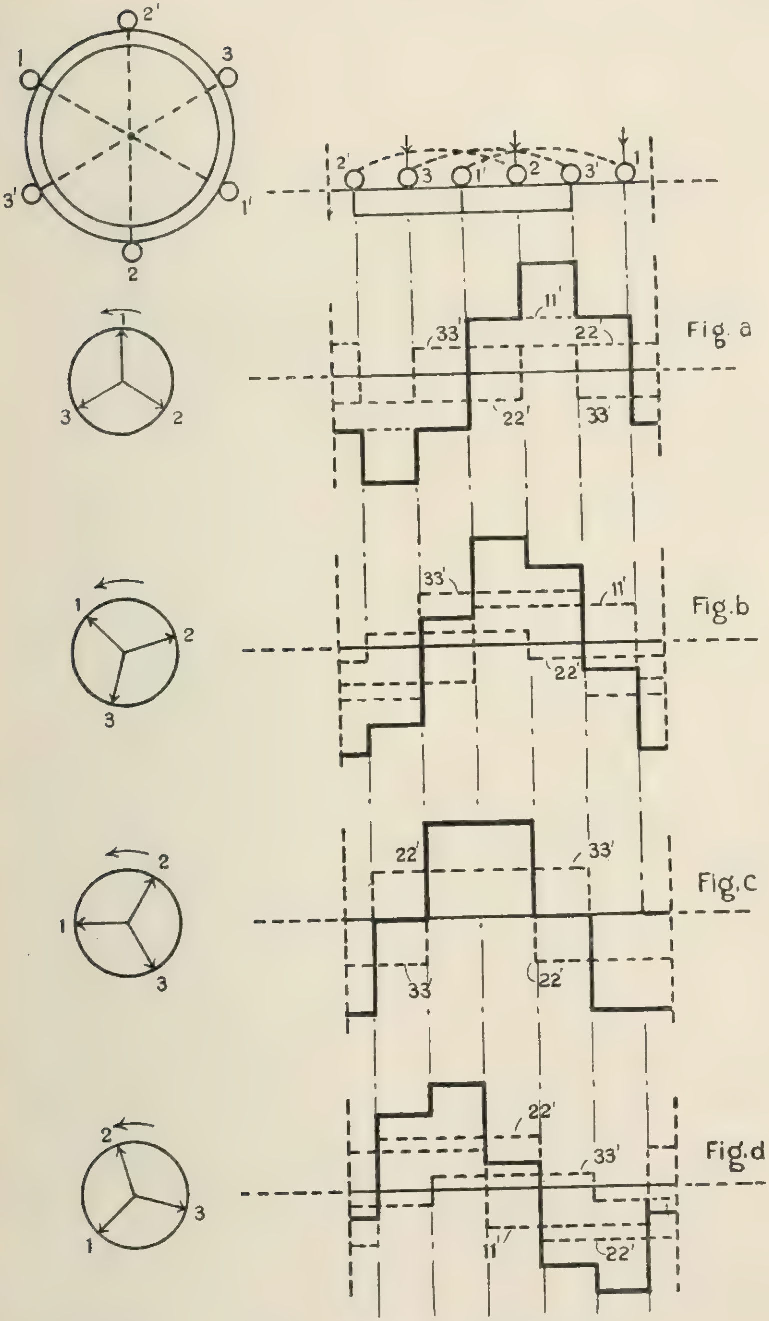
Suppose alternating currents be supplied to the respective phase windings, so that—

$$i_1 = \bar{I} \sin \omega t : i_2 = \bar{I} \sin (\omega t - \frac{2}{3}\pi) : i_3 = \bar{I} \sin (\omega t - \frac{4}{3}\pi).$$

At the instant $t = \frac{\pi}{2\omega}$ the current in phase I will have its maximum positive value, and the currents in phases II and III will each be negative and half the maximum value. The corresponding magnetic fields will be as shown dotted in Fig. 403*a*, and the resultant field, found by adding the corresponding ordinates, is shown by the full-line figure. In Fig. 403 (*b–d*) are shown the field distribution for the times $t = \frac{3\pi}{4\omega} : \frac{\pi}{\omega} : \frac{5\pi}{4\omega}$. It will be seen that the resultant field changes its wave form from moment to moment and moves from right to left—*i.e.*, rotates in a counter-clockwise direction.

If similar diagrams be drawn for a complete period of the current it will be found that the resultant field rotates at a uniform angular speed ω.

If instead of considering the rectangular waves due to the current in each phase the fundamental sine wave in each case is considered, and a series of waves of field distribution at the times $t = \frac{\pi}{2\omega} : \frac{3\pi}{4\omega} : \frac{\pi}{\omega} : \frac{5\pi}{4\omega}$ respectively as shown in Fig. 404 (*a–d*), it will be found that *the resultant field is a sine wave of constant amplitude which is one and a half times the maximum amplitude of each individual sine wave.* It will also be seen that the resultant sine wave rotates with uniform angular velocity ω. This same result can be arrived at by resolving the alternating field due to the current in each phase into two oppositely



Vector Diagram of Currents. Magnetic Fields.

Fig. 403.

rotating fields, each of which has half the amplitude of the amplitude of the alternating field, as explained in § 263.

The method here described of producing a rotating field by means of a three-phase, alternating current winding without the mechanical rotation of the core is of great practical importance, and is the principle of action of the induction motor.

If the supply to *any two* of the phases is interchanged, say the current $i_2=\bar{\mathrm{I}}\sin\left(\omega t-\frac{2\pi}{3}\right)$, supplied to phase 33^1, and the current $i_3=\bar{\mathrm{I}}\sin\left(\omega t-\frac{4\pi}{3}\right)$ supplied to phase 22^1, it will be found by drawing diagrams similar to those of Fig. 404 that the *direction of rotation of the resultant field is reversed.*

Not only do the three fundamental sine waves of the three phases combine to produce a rotating field, but the corresponding harmonics of the three phases also combine to produce rotating fields as follows:

If the fundamental resultant wave rotates in a *counter-clockwise* direction, as in Fig. 404, then—

The resultant due to the *third* harmonic is *zero.*

The resultant due to the *fifth* harmonic rotates *clockwise.*

The resultant due to the *seventh* harmonic rotates *counter-clockwise.*

The resultant due to the *ninth* harmonic is *zero.*

The resultant due to the *eleventh* harmonic rotates *clockwise.*

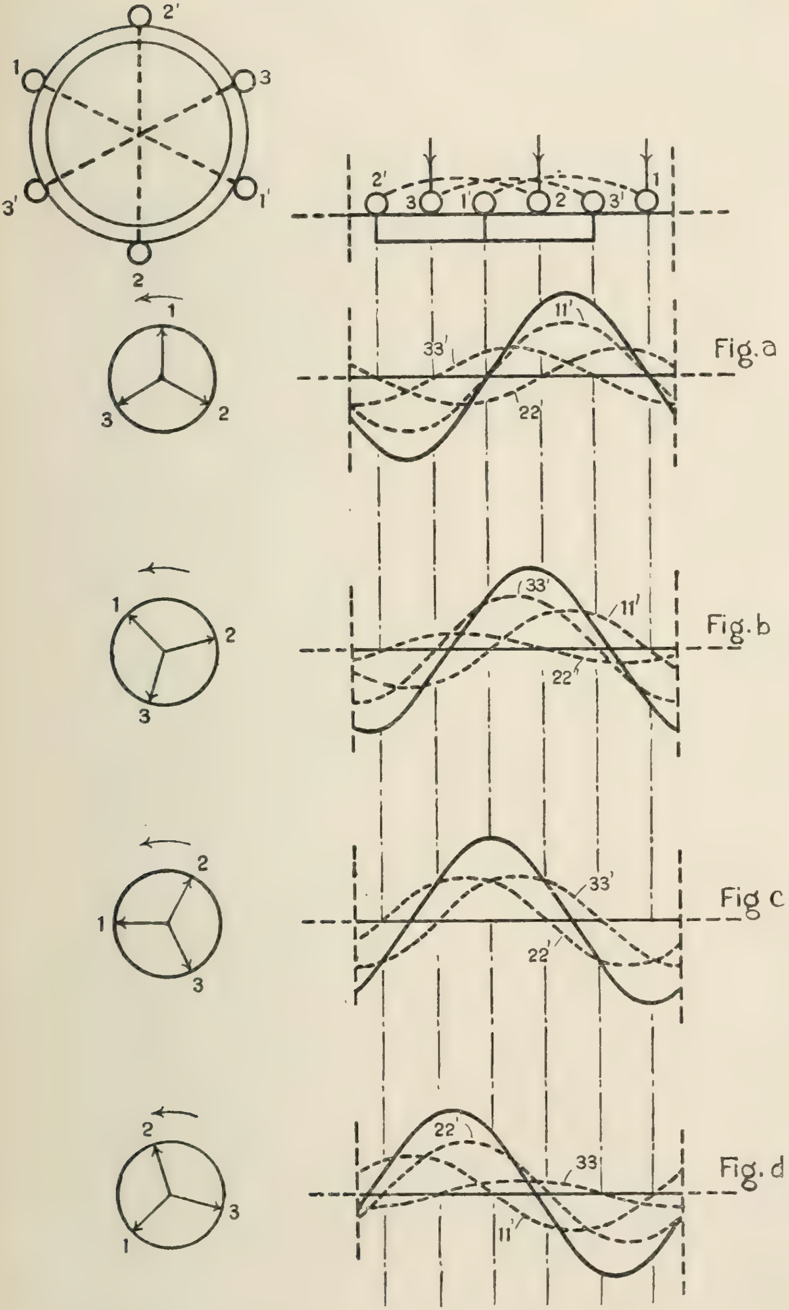
The resultant due to the *thirteenth* harmonic rotates *counter-clockwise.*

The resultant fields due to all whole multiples of the third harmonic of the flux distribution waves are zero.

If the resultant field for the xth harmonic of the individual field distribution curves be traced out by means of a series of diagrams similar to those in Fig. 404 (*a–d*), it will be found that the speed of the rotating field due to the xth harmonic will be $\frac{1}{x}$ of the speed of the rotating field due to the fundamental wave.

The winding factors for three-phase concentrated windings can be immediately deduced from Table I. given on p. 407. The quantity Q has the same meaning as in Table I., but for a three-phase winding $q=\frac{\mathrm{Q}}{3}$—*i.e., the number of holes per pole per phase.* Thus, if there are six holes per pole—*i.e.*, $\mathrm{Q}=6$: $q=2$—the winding factors are—

$$f_{w1}=0{\cdot}966,$$
$$f_{w3}=0{\cdot}707,$$
$$f_{w5}=0{\cdot}259.$$



Vector Diagrams of Currents. Magnetic Fields.

Fig. 404.

The winding factor enables the amplitude of the alternating field per phase to be obtained (see also § 264), the fields due to the three phases combining to form the rotating field.

It is to be observed that the results obtained are independent of whether the three-phase winding is star or mesh connected (see §§ 240, 241), the field due to each phase being calculated from the current in that phase.

The remarks on the winding factors of distributed single-phase windings (§ 265) can be immediately applied to the case of a three-phase winding, and the winding factors given in Table II., p. 408, apply to three-phase windings if the angle $\frac{S}{\tau}\pi$ be taken to be the angle subtended at the centre of the core by the winding of each phase.

The results hold for the case of windings with p pairs of poles if the angles θ, Figs. 399–401 and a, Fig. 402, be in electrical measure (see p. 345).

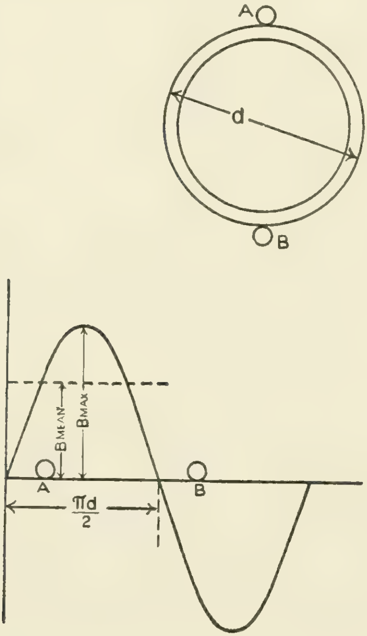


Fig. 405.

From what has been said it will be seen that if a three-phase, two-pole winding be supplied with three-phase current of frequency f cycles per second the speed of the resultant fundamental rotating magnetic field will be f revolutions per second.

If the machine has *p pairs* of poles the resultant rotating field will make—

$$\frac{f}{p} \text{ revolutions per second.}$$

267. The E. M. F. induced in an Alternating Current Winding.—Consider first a coil of one turn wound in a pair of diametrically opposite slots of the outer iron core, as shown in Fig. 405, and assume that a sine wave of magnetic flux is, by some means, produced in the air-gap between the two concentric iron cores. Let the magnetic field rotate relatively to the coil with an angular velocity ω. It is a matter of indifference whether the relative velocity is due to a stationary coil and rotating field, or a stationary field and rotating coil, or both coil and field rotating at different speeds.

The e. m. f. induced at any moment in each coil side is (see p. 344)—

$$B_x lv 10^{-8} \text{ volts.}$$

where B_x is the flux density of the field cutting the coil side at that instant, v is the relative velocity of the field and coil side in cms. per second, and l cms. is the length of the coil side. The e. m. f.'s induced in the two sides of the coil act in the same direction round the coil, and hence the e. m. f. induced in the coil at any instant is—

$$e = 2B_x lv 10^{-8} \text{ volts.}$$

If d cms. is the diameter of the core and n the relative speed of rotation of coil and field in revolutions per second—

$$\omega = 2\pi n,$$

$$v = \pi dn = \pi df,$$

where f is the frequency of the induced e. m. f. in cycles per second.

Hence $$e = 2B_x l\pi df 10^{-8} \text{ volts.}$$

The maximum value of the e. m. f. is—

$$\bar{E} = 2B_{max} l\pi df 10^{-8} \text{ volts.}$$

Let Φ represent the flux per pole in the air-gap.

Then $$\Phi = B_{mean} \frac{\pi d}{2} l.$$

But $$B_{mean} = \frac{2}{\pi} B_{max} \text{ (p. 339).}$$

Therefore $$\Phi = dlB_{max}.$$

Hence $$\bar{E} = \Phi 2\pi f 10^{-8} \text{ volts.}$$

The r. m. s. value of the e. m. f. is (p. 338)—

$$E = \frac{\bar{E}}{\sqrt{2}} = \pi \sqrt{2} \Phi f 10^{-8} \text{ volts;}$$

or $$E = 4{\cdot}44 \Phi f 10^{-8} \text{ volts.}$$

If there are w turns in the coil the r. m. s. value of the induced e. m. f. is—

$$E = 4{\cdot}44 \Phi f w 10^{-8} \text{ volts.}$$

If instead of the winding of w turns being concentrated in a single pair of holes, as in Fig. 405, it is concentrated in q pairs of holes, equally spaced—that is, there are $\frac{w}{q}$ turns per pair of holes—the e. m. f. induced in the windings in consecutive pairs of holes will be relatively displaced in phase, and the resultant e. m. f. will therefore be less than the arithmetical sum of the e. m. f.'s induced in the winding in each pair of holes.

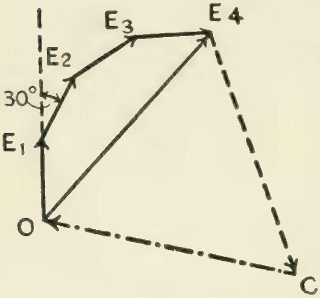


Fig. 406.

For example, if there are two pairs of holes displaced by 30°, and if the e. m. f. induced in the winding in one pair be given by the vector OE_1 (Fig. 406), and that in the other pair by the vector E_1E_2, the resultant e. m. f. for the whole winding will be given by the vector OE_2. But the ratio $\frac{OE_2}{2E_1E_2}$ is equal to the *winding factor* as defined in § 264.

Hence, if the winding factor is f_{w1}, and the e. m. f. induced in the winding is equally distributed in q pairs of holes equally spaced, and if E^1 is the e. m. f. induced in the turns in one pair of holes, the total e. m. f. will be—

$$E = f_{w1} q E^1.$$

But from the previous formula—

$$E^1 = 4{\cdot}44 \Phi f \frac{w}{q} 10^{-8} \text{ volts.}$$

Hence $$E = 4{\cdot}44 f_{w1} \Phi f w 10^{-8} \text{ volts.}$$

The value of the winding factor f_{w1} for different types of windings may be taken from the Tables I. and II. (pp. 407 and 408).

If the flux wave have $2p$ poles, the above formula is still applicable if w is the total number of turns in series per phase and Φ the flux per pole. The frequency in this case will be—

$$f = np.$$

where n is the relative speed in revolutions per second of field and winding.

If the field is not a rotating field, but a pulsating field of frequency f (for example, the field may be due to an alternating current flowing in the winding, Fig. 405), it can be shown that the r. m. s. value of the e. m. f. induced in the winding is given by—

$$E = 4{\cdot}44 f_{w1} \Phi f w 10^{-8} \text{ volts,}$$

where Φ is the maximum value of the flux per pole, and the other symbols have the same significance as before (see p. 433).

268. The Magnetomotive Force and the Magnetising Current of an Alternating Current Winding.—In the following treatment the problem is considered for the case in which the winding is arranged in holes near the surface of an iron core, as shown in Figs. 396 and 407, and a concentric iron core forms the completion of the magnetic circuit, there being an annular air-gap between the two smooth surfaces of the iron cores. It is further assumed that the permeability of the iron cores is very great, so that the magnetomotive force necessary to drive the flux through the iron is negligibly small (see p. 267).

The iron losses due to the alternating magnetic flux are also neglected, and the pressure drop due to the resistance of the winding (IR drop) is assumed to be negligibly small as compared with the induced e. m. f. of the winding.

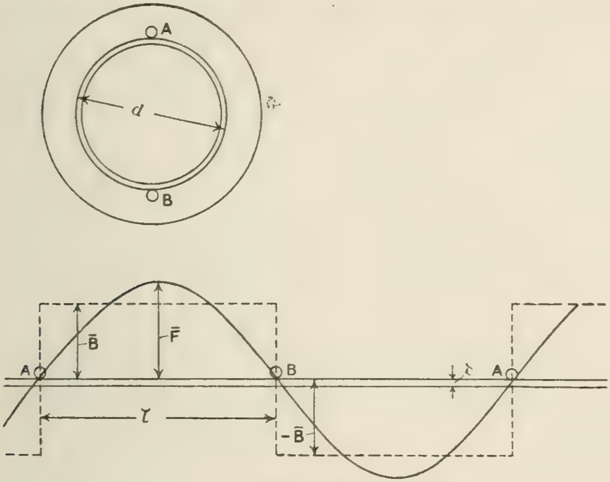


Fig. 407.

These assumptions are permissible for many practical purposes as a first approximation.

Case I. Single-Phase Winding concentrated in one Slot Per Pole.—The winding is arranged in the holes A and B in the outer core (Fig. 407), and thus forms a two-pole winding. In Fig. 407 is also shown the development of the periphery of the annular air-gap between the inner and outer core.

If a current flows in the winding concentrated in the holes A and B the flux density in the air-gap may be represented by a rectangular figure, as explained in § 264.

This rectangle may be analysed into a series of sine waves (see § 262, Fig. 394), the fundamental of which is shown in Fig. 407, the higher harmonics being neglected in what follows.

Fig. 407 is drawn for the moment at which the current in the winding has its maximum value, the current being assumed to be alternating and to vary as a sine function of the time.

The *maximum value* of the flux per pole is therefore given by the area of the half-wave AB (Fig. 407)—that is,

$$\Phi = \overline{\mathrm{F}}\tau \frac{2}{\pi} l,$$

where l cms. is the length of the core in the direction perpendicular to the plane of the paper.

The r. m. s. value of the e. m. f. induced in the winding is therefore (p. 413)—

$$\mathrm{E} = \frac{4{\cdot}44\Phi wf}{10^8} \text{ volts},$$

where w is the total number of turns in series in the winding and f the frequency of the alternating current in the winding.

Substituting the above expression for Φ in the formula for the e. m. f. gives—

$$\mathrm{E} = \frac{2{\cdot}83\overline{\mathrm{F}}\tau lwf}{10^8} \text{ volts}.$$

The maximum value of the magnetomotive force of the winding (Fig. 407) is—

$$\frac{4\pi}{10} w \overline{\mathrm{I}}$$

$$= \frac{4\pi}{10} w\sqrt{2}\,\mathrm{I},$$

where $\overline{\mathrm{I}}$ is the maximum value of the current and I the r. m. s. value.

Hence, from p. 267,

$$\frac{4\pi}{10} w\sqrt{2}\,\mathrm{I} = \overline{\mathrm{B}}2\delta,$$

where δ cms. is the length of the air-gap (Fig. 407). But by § 262—

$$\overline{\mathrm{F}} = \frac{4}{\pi} \overline{\mathrm{B}}.$$

Therefore $$\mathrm{I} = \frac{0{\cdot}9\overline{\mathrm{F}}\delta}{w} \text{ ampères};$$

that is, the r. m. s. value of the magnetising current is—

$$\mathrm{I} = \frac{0{\cdot}314\mathrm{E}10^8\delta}{w^2\tau lf} \text{ ampères}.$$

If the winding forms $2p$ poles (see Fig. 408, which represents a four-pole winding), and if there are w turns in series in the whole winding—

$$E=\frac{2\cdot83\bar{F}\tau l w f}{10^8}\text{ volts, where }\tau=\frac{\pi d}{2p}.$$

Also $$\frac{4\pi}{10}\frac{w}{p}\sqrt{2}I=\bar{B}2\delta.$$

Hence $$I=\frac{0\cdot314Ep10^8\delta}{w^2\tau l f}\text{ ampères.}$$

Case II. Single-Phase Winding concentrated in q Equally Spaced Slots per Pole.—In Fig. 409 is shown the development of such a winding. Let w be the total number of turns distributed in q equally spaced holes per pole (in Fig. 409 $q=2$), and let there be $2p$ poles. It will be seen from the results already deduced (see § 265) that if $\bar{F}$ is the maximum amplitude of the fundamental wave of the resultant field and $\bar{F}_1$ the maximum amplitude of the fundamental wave of the field due to the windings concentrated in one hole per pole, then

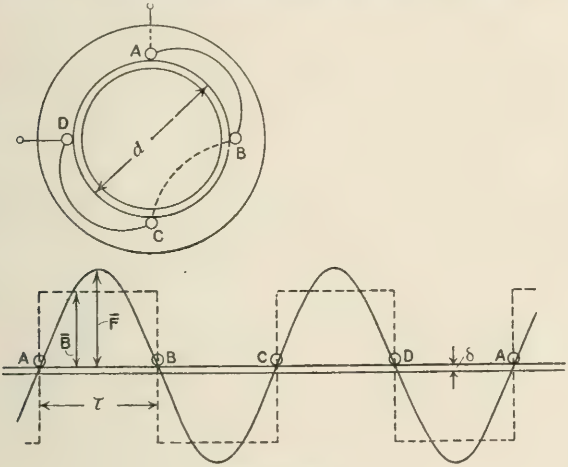


FIG. 408.

$$\bar{F}=qf_{w1}\bar{F}_1,$$

where f_{w1} is the winding factor of the fundamental wave (see pp. 363 and 407).

Further, the r. m. s. value of the resultant e. m. f. E induced in the whole winding is given by the expression (pp. 363 and 414)—

$$E=qf_{w1}E_1\text{ volts,}$$

where E_1 volts is the r. m. s. value of the e. m. f. induced in the part of the winding concentrated in one hole per pole—that is, the e. m. f. induced in the part of the winding concentrated in the holes AB (Fig. 409).

The r. m. s. value of the magnetising current I is given by the equation—

$$\frac{4\pi}{10}\frac{w}{pq}\sqrt{2}\,I=\frac{\pi}{4}\bar{F}_1 2\delta$$

as before (see Case I. above and Fig. 407).

Hence $$I=0{\cdot}89\frac{\bar{F}_1 pq\delta}{w}\text{ ampères,}$$

or $$I=0{\cdot}89\frac{p\delta}{wf_{w1}}\bar{F}\text{ ampères.}$$

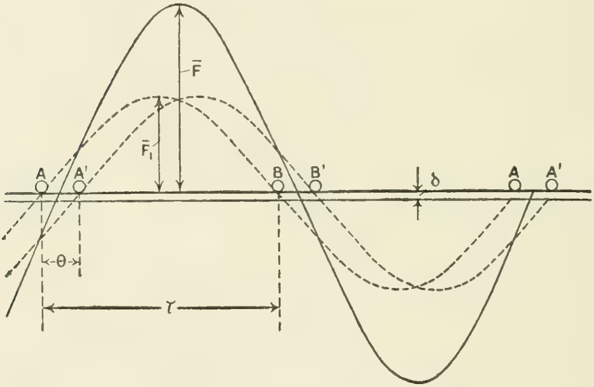


Fig. 409.

Also $$E=\frac{4{\cdot}44\Phi wff_{w1}}{10^8}\text{ volts,}$$

where $$\Phi=\frac{2}{\pi}\bar{F}\tau l.$$

Hence $$E=\frac{4{\cdot}44\tau lwff_{w1}}{10^8}\frac{2}{\pi}\bar{F};$$

that is, the r. m. s. value of the e. m. f. induced in the winding is—

$$E=\frac{4{\cdot}44\tau lw^2ff^2_{w1}}{10^8\times0{\cdot}89\delta p}\frac{2}{\pi}I\text{ volts,}$$

and the r. m. s. value of the magnetising current is—

$$I=\frac{0{\cdot}314Ep10^8\delta}{w^2\tau lff^2_{w1}}\text{ ampères.}$$

Case III. Three-Phase Winding.—If $\bar{F}$ is the maximum amplitude of the fundamental sine wave of flux distribution due to one phase, the resultant field due to the three phases will have a fundamental sine wave of constant amplitude $\bar{F}_R$, and rotating at a uniform speed in space, as shown in p. 408 and Fig. 404.

Also
$$\bar{F}_R = 1{\cdot}5\bar{F}.$$

The results deduced in Case II. may be immediately applied to this Case.

Thus
$$\bar{F} = \frac{wf_{w1}}{0{\cdot}89p\delta}\,I,$$

where f_{w1} is the winding factor for one phase and I ampères is the r. m. s. value of the current in one phase.

The r. m. s. value of the induced e. m. f. per phase is—

$$E = \frac{4{\cdot}44\tau lwff_{w1}}{10^8}\,\frac{2}{\pi}\,\bar{F}_R \text{ volts},$$

where w is the number of turns per phase of the winding.

The r. m. s. value of the magnetising current per phase is—

$$I = \frac{0{\cdot}89p\delta}{wf_{w1}}\,\bar{F} \text{ ampères};$$

that is,
$$I = \frac{0{\cdot}593p\delta}{wf_{w1}}\,\bar{F}_R \text{ ampères},$$

since
$$\bar{F}_R = 1{\cdot}5\bar{F}.$$

Hence the r. m. s. value of the magnetising current per phase is—

$$I = \frac{0{\cdot}21Ep10^8\delta}{w^2\tau lff^2_{w1}} \text{ ampères}.$$

269. Inductance of a Single-Phase Transmission Cable.—In Fig. 410 let the two cables of a single-phase transmission line be represented by the two circles A and B distant a cms. apart, the diameter of each cable being d cms. and the length l cms.

The intensity H_x of the magnetic field at a point distant x cms. from the centre of, say, cable A, when a current of 1 ampère flows in the cable, is given by the expression (see p. 263)—

$$H_x 2\pi x = \frac{4\pi}{10};$$

that is,
$$H_x = \frac{0{\cdot}2}{x}$$

for values of x greater than $\frac{d}{2}$; that is, for points outside the cable.

If this value of H_x be plotted as a function of x a rectangular hyperbola will be obtained as shown in Fig. 410, where the distances x are measured along AB.

When a current of 1 ampère flows in cable A in a direction, say, away from the observer, a current of 1 ampère will flow in cable B in a direction towards the observer, since cable B forms the completion of the circuit of cable A. The intensity of the magnetic field outside cable B will also be a rectangular hyperbola, and the two rectangular hyperbolæ for the cables A and B respectively will have the relative positions shown in Fig. 410.

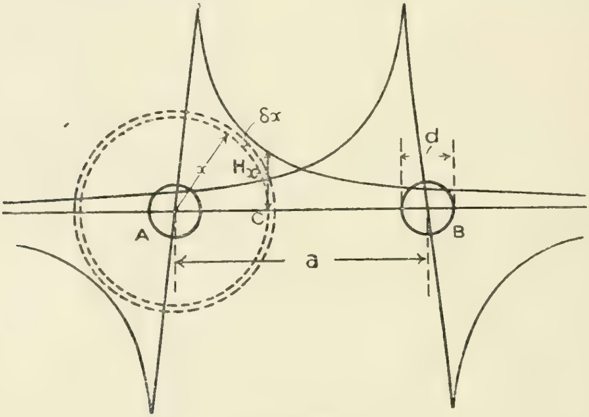


Fig. 410.

The formula given above for the intensity of the magnetic field does not hold for values of x less than $\frac{d}{2}$; that is, for points inside the cable.

For example, suppose the cable is divided up into coaxial cylindrical filaments. It can be shown by an application of Laplace's formula (§ 186) that for any of the cylindrical filaments the current in that filament produces no magnetic flux within the filament. This also follows from the result of § 192. The flux linking the filaments near the centre of the conductor is therefore greater than the flux linking the filaments near the surface of the conductor. In other words, the inductance of the inner filaments is greater than the inductance of the outer filaments.

Assuming the current density throughout the cross-section of the conductor to be constant, the intensity of the magnetic field at points *inside* the conductor for 1 ampère flowing is given by $H^1{}_x$ in the formula—

$$2\pi x H^1{}_x = \frac{4\pi}{10} \frac{x^2}{\left(\frac{d}{2}\right)^2},$$

where x is less than $\frac{d}{2}$.

Hence $$H^1{}_x = 0\cdot8 \frac{x}{d^2},$$

and the intensity of the magnetic field at a point inside the conductor is therefore directly proportional to the distance of the point from the centre of the conductor.

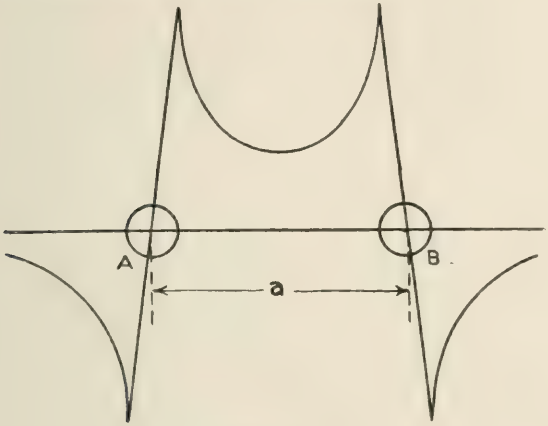


Fig. 411.

The values of $H^1{}_x$ so obtained and plotted as a function of x gives a straight line, as shown in Fig. 410, in which the distances x are measured along the line AB.

Each of the cables A and B will produce a similar field, and the total intensity of the field between the two cables is given by adding corresponding ordinates of the curves shown in Fig. 410. The resultant intensity so found is shown in Fig. 411.

The coefficient of self-induction L_1 henrys of each cable is given by—

$$\tfrac{1}{2}L_1 i^2 = \text{energy of the magnetic field in joules},$$

where i ampères is the current in the cable.

The energy in joules of the magnetic field may also be written—

$$\frac{1}{2\times10^8}\Sigma\,[\text{flux} \times \text{ampères linked by the flux}].$$

If the current in the cable is 1 ampère—

$$L_1=\frac{1}{10^8}\left[\sum_{x=\frac{d}{2}}^{x=a} H_x dx+\sum_{x=\frac{d}{2}}^{x=o} H^1{}_x\frac{x^2}{\left(\frac{d}{2}\right)^2}dx\right] \text{ henrys per cm. length,}$$

neglecting the small portion of the field which extends beyond $x=a$.

By substituting the expressions already found for H_x and $H^1{}_x$, the coefficient of self-induction of *each cable* is given by—

$$L_1=\frac{l}{10^3}\left[0{\cdot}74 \log_{10}\left(\frac{2a}{d}\right)+0{\cdot}08\right] \text{ henrys}$$

for l in miles.

For the *two cables* A and B in series the coefficient of self-induction is—

$$L=2L_1=\frac{2l}{10^8}\left[0{\cdot}46 \log_{10}\left(\frac{2a}{d}\right)+0{\cdot}05\right] \text{ henrys}$$

for l in cms.;

that is— $$L=\frac{l}{10^3}\left[1{\cdot}48 \log_{10}\left(\frac{2a}{d}\right)+0{\cdot}16\right] \text{ henrys,}$$

where l is the distance of transmission in miles.

In the above determination of the inductance L the assumption has been made that the current density is the same throughout the cross-section of the cable. When an alternating current passes through the cable a uniform current density would imply a greater induced e. m. f. in the central filaments of the cable than in filaments near the surface, since the flux linkage of the central filaments would be greater than that of the outer layer, as already explained. The result is that the current distribution is altered so that the density in the outer filaments of the cable is greater than the current density in the inner filaments. In other words, the current tends to concentrate near the surface of the conductor, and this effect is more marked the greater the frequency of the current in the cable (see also p. 425 and Fig. 422).

The concentration of the current towards the surface of the cable is known as the **skin effect,** and causes an increase of the effective resistance of the cable when carrying alternating current as compared with the resistance when carrying direct current (see § 272).

270. Inductance of a Single-Phase Concentric Cable.—In the case of a concentric cable as shown in Fig. 412, the flux linking the outer cable is zero, since the flux due to the current in the inner core will neutralise the flux due to the same current flowing in the opposite direction in the outer core for all points outside the outer core. The inductance of the concentric single-phase cable is therefore—

$$L = \frac{l}{10^3}\left[0{\cdot}74 \log_{10}\left(\frac{2a}{d}\right) + 0{\cdot}08\right] \text{ henry,}$$

where l is the distance of transmission in miles.

This formula for L is deduced in a similar manner to that in § 269.

The skin effect for such a cable will only be appreciable for the inner core.

Owing to the relatively large electrostatic capacity between the two concentric cores (see p. 49), this type of cable is, for many purposes, not very satisfactory.

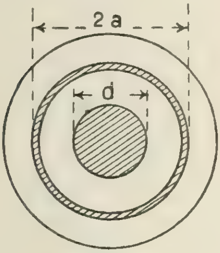


FIG. 412.

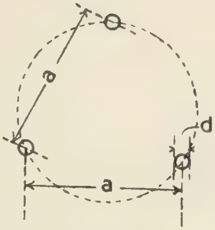


FIG. 413.

271. Inductance of a Three-Phase Transmission System.—In this case (Fig. 413) any two cables may be considered as the return for the third cable, and the coefficient of self-induction of each line is the same as that for each line of the single-phase system of Fig. 410.

That is, $$L = \frac{1}{10^3}\left[0{\cdot}74 \log_{10}\left(\frac{2a}{d}\right) + 0{\cdot}08\right] \text{ henrys per mile.}$$

272. The Resistance of Transmission Cables to Alternating Currents—Skin Effect.—It was pointed out in § 269 that when an alternating current is flowing in a cable the current tends to concentrate near the surface, and the effective resistance of the conductor is consequently increased as already stated. This concentration of the current towards the surface of the conductor is known as the *skin effect.*

Let f cycles per second be the frequency of the current.

Let d cms. be the diameter of the cable.

Let ρ be the specific resistance of the material of the conductor in ohms. per cm. cube.

And let $$Y = \frac{fd^2}{\rho 10^9}.$$

For hard-drawn copper $10^9\rho$ may be taken as 1620 (see p. 194),

so that $$Y \text{ becomes } \frac{fd^2}{1620}.$$

Then for values of Y less than about 0·5, the effective resistance of a cable to alternating current may be written for a non-magnetic conductor—

$$R_{eff}=R\,[1+2Y^2-3{\cdot}3Y^4],$$

where R is the resistance measured by direct current.

In the following table the *percentage increase* of R for hard-drawn copper conductors is given for different values of *d* and *f*.

Frequency f =	25	50	100	250	500
d = 0·8 cm.	—	—	—	2	7·5
d = 1·2 cms.	—	—	—	8·9	26·4
d = 1·5 cms.	—	—	3·8	19·6	—
d = 2 cms.	—	3	11·1	—	—
d = 3 cms.	3·8	13·7	30	—	—

Thus, for a cable 8 mm. diameter and carrying an alternating current at a frequency of 500 cycles per second, the effective resistance is about 1·08 times the resistance measured by direct current.

For values of Y greater than about 1 the above formula does not hold, and in such cases the following formula may be used—viz.:

$$R_{eff}=R\,[1{\cdot}6\sqrt{Y}+0{\cdot}25].$$

Thus, if $f=10^6$ cycles per second: $d=1$ mm.: $Y=6{\cdot}2$:

$$R_{eff}=4{\cdot}23R.$$

If the two cables of a single-phase transmission system are relatively close together the flux distribution in the cable itself is no longer given by cylindrical curves concentric with the centre of the cable, but the fluxes due to each cable mutually react, and the current tends to concentrate to one side of the cable, the effective resistance being consequently appreciably increased.

For high-frequency currents of large magnitude—for example, for lightning arrester connections—a thin, flat copper conductor offers a very much smaller resistance than a round wire of the same cross-section.

273. Eddy Current and Hysteresis Losses in Iron.—When an alternating magnetic flux is produced in iron the variation of the flux produces *eddy currents* in the iron in planes at right angles to the direction of the flux, the current being such that they tend to prevent the change of magnetic flux producing them (see § 195).

In Fig. 414 is shown the cross-section of a bar of iron, and it is assumed that a sinusoidal alternating magnetic flux is produced in a direction perpendicular to the plane of the paper.

Consider an elementary closed band *abcd* of the iron section. The flux threading this band will produce an e. m. f. acting round the band which will lag by $\frac{1}{4}$ period on the flux (see Fig. 414). The vector $O\Phi_m$ of the flux is drawn in Fig. 414 horizontally. If the instantaneous value of this flux is $\Phi_m \sin \omega t$, the e. m. f. induced in the band *abcd* will be—

$$e = -\frac{d\,(\Phi_m \sin \omega t)}{10^8 dt} = -k\Phi_m\omega \cos \omega t \text{ volts (see p. 272).}$$

or $$e = k\omega\Phi_m \sin\left(\omega t - \frac{\pi}{2}\right) = \overline{E} \sin\left(\omega t - \frac{\pi}{2}\right) \text{ volts.}$$

This expression for e shows that the vector of the induced e. m. f. lags by 90° (*i.e.*, $\frac{1}{4}$ period) on Φ_m.

The e. m. f. e will produce a current i in the band *abcd* lagging by an angle ϕ on OE, the value of ϕ depending on the resistance and self-induction of the band *abcd*. The current i will produce a corresponding flux shown by the vector ϕ_i, which will be in phase with the current vector OI, and the actually existing flux threading the band *abcd* will therefore be given by the vector ϕ, which is less than ϕ_m. The effect of the eddy currents therefore is to diminish the flux threading the band *abcd* (see also § 269 and Fig. 422).

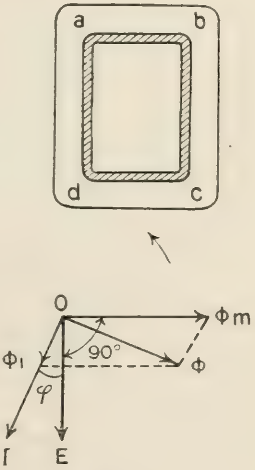


FIG. 414.

Consider the cross-section of a plate of iron in which an alternating flux is produced in the direction perpendicular to the cross-section. Let Fig. 415 represent a narrow band of thickness δx cms. and breadth $2x$ cms. If y cms. be the length of the band the area enclosed by the band will be $2xy$ sq. cms.

The r. m. s. value of the e. m. f. induced in the band by the alternating flux will be—

$$E = kf\,B_{max}\,2xy \text{ volts,}$$

where k is a constant, f is the frequency, and B_{max} the maximum value of the flux density through the band during the cycle of flux alternation.

The resistance of the band per cm. length measured perpendicular to the plane of the paper will be—

$$R = \frac{k_1(2y+4x)}{\delta x} \text{ ohms,}$$

where k_1 is the specific resistance of the iron.

The r. m. s. value of the current induced in the band will therefore be—

$$I = \frac{k_2 f B_{max} \, 2xy\delta x}{2y+4x} \text{ ampères;}$$

or, taking the practical case in which x is small compared with y, so that $2y+4x$ may be taken as very approximately equal to $2y$—

$$I = k_2 f B_{max} \, x\delta x \text{ ampères,}$$

where k_2 is a constant.

The power loss is—

$$I^2R = k_3(f B_{max})^2 x^2 y \delta x \text{ watts.}$$

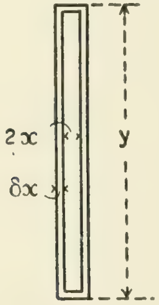


FIG. 415.

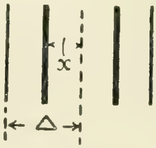


FIG. 416.

If 2Δ cms. is the total width of the cross-section of the iron (Fig. 416), the total power loss due to eddy currents is—

$$\int_0^{\Delta} k_3(f B_{max})^2 x^2 y dx \text{ watts}$$

$$= k_3(f B_{max})^2 y \int_0^{\Delta} x^2 dx \text{ watts}$$

$$= k_3(f B_{max})^2 y \frac{\Delta^3}{3} \text{ watts,}$$

it being assumed that the breadth of the iron section 2Δ (Fig. 416) is small compared with the length, so that y may be taken as approximately the same for all the elementary bands such as that of Fig. 415.

But the volume of the plate per cm. length perpendicular to the plane of the paper is—

$$2\Delta y \text{ cubic cms.}$$

Therefore the total power loss is—

$$k_4(f B_{max} \Delta)^2 \text{ watts per cubic cm.,}$$

where 2Δ cms. is the width of the iron section.

The eddy current loss in the iron may therefore be written—

$$W_e = \epsilon(f B_{max} \delta)^2 \text{ ergs per second per c.cm.,}$$

where δ cms. is the thickness of the iron section.

It follows from the formula on p. 140 that the hysteresis loss in the iron may be represented by the formula—

$$W_h = \eta f B_{max}^{1\cdot6} \text{ ergs per second per c.cm.}$$

Strictly η is not a constant but depends upon the frequency f. The variation of η, however, over the range of frequencies usually met with in electrical engineering may be neglected.

The total iron loss may therefore be written—

$$W = W_h + W_e = [\eta f B_{max}^{1\cdot6} + \epsilon(f B_{max} \delta)^2] \text{ ergs per second per c.cm.;}$$

or— $$W = f[\eta B_{max}^{1\cdot6} + \epsilon f (B_{max} \delta)^2] \text{ ergs per second per c.cm.}$$

The coefficients ϵ and η may be experimentally determined as follows:

If measurements of the power loss W and the frequency f be made for various values of the frequency and for a constant value of the induction density B_{max}, and if the values of $\frac{W}{f}$ be plotted as a function of f (see Fig. 418), it follows from the above expression for W that the points will lie on a straight line, which will cut the axis of $\frac{W}{f}$ such that the intercept on this axis will be equal to $\eta B_{max}^{1\cdot6}$, and hence the value of η may be obtained. Further, the line will be inclined to the axis of f at an angle θ, such that—

$$\tan \theta = \epsilon(B_{max} \delta)^2,$$

and hence ϵ may be determined.

For good transformer iron—

$$\eta = 0\cdot00135 \text{ and } \epsilon = 1\cdot5 \times 10^{-4}.$$

Hence the iron loss may be written—

$$W = [0\cdot00135 f B_{max}^{1\cdot6} + 1\cdot5 \times 10^{-4}(f B_{max} \delta)^2] \text{ ergs per second per c.cm.}$$

$$= 59 \times 10^{-7}[0\cdot00135 f B_{max}^{1\cdot6} + 1\cdot5 \times 10^{-4}(f B_{max}\delta)^2] \text{ watts per lb.,}$$

where δ cms. is the thickness of the plates.

A more convenient way of expressing this is—

$$W = 0{\cdot}048\left[\frac{f}{100}\left(\frac{B_{max}}{1000}\right)^{1{\cdot}6} + 180\left(\frac{f}{100}\right)^2\left(\frac{B_{max}}{1000}\right)^2\delta^2\right]\text{watts,}$$

per lb.

where B_{max} is the maximum value of the alternating flux density, δ cms. is the thickness of the iron, and f cycles per second is the frequency of the alternating flux.

274. High Resistance Iron Alloys.—For some years special iron alloys have been manufactured for use in transformers and in other cases in which it is desired to keep the iron losses as low as possible. One of the best known of these alloys is "stalloy" (see also p. 141 and § 115), the composition of which comprises among other ingredients a small percentage of silicon. The specific resistance of this alloy is much higher than that of ordinary transformer plates (see p. 194), and consequently the eddy current loss is lower. The hysteresis loss for this alloyed iron is also lower than for ordinary transformer plates.

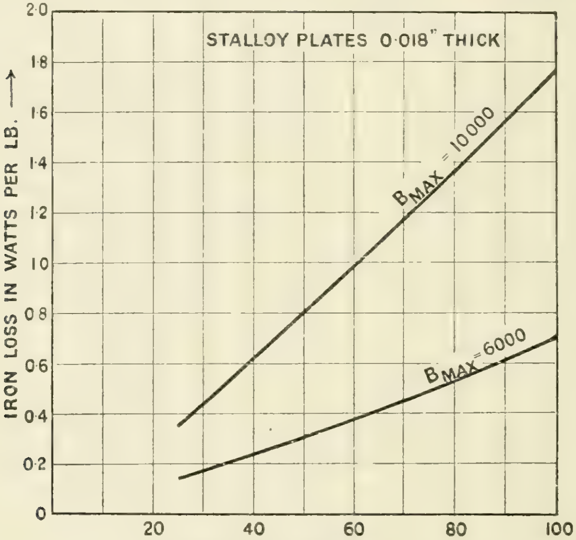


Fig. 417.

In Fig. 417 are given two curves of the total loss in watts per lb. for stalloy plates 0·018″ (*i.e.*, 0·45 mm.) thick. The loss is plotted as a function of the frequency for values of B_{max} 6000 and 10000 respectively.

If the loss is divided by the frequency, as explained on p. 427, and the joules per lb. per cycle so obtained plotted as a function of the frequency, the straight lines shown in Fig. 418 are obtained. From these results it is found that the iron loss for stalloy plates is given by—

$$W = 0{\cdot}034\left[\frac{f}{100}\left(\frac{B_{max}}{1000}\right)^{1{\cdot}6} + 58\left(\frac{f}{100}\right)^2\left(\frac{B_{max}}{1000}\right)^2\delta^2\right] \text{ watts per lb.,}$$

where δ cms. is the thickness of the plates.

In Figs. 419 and 420 the iron losses for stalloy are shown for a frequency of 50 cycles per second, and for various values of the maximum induction density. In the same figures the losses are shown for good quality transformer iron "lohys"

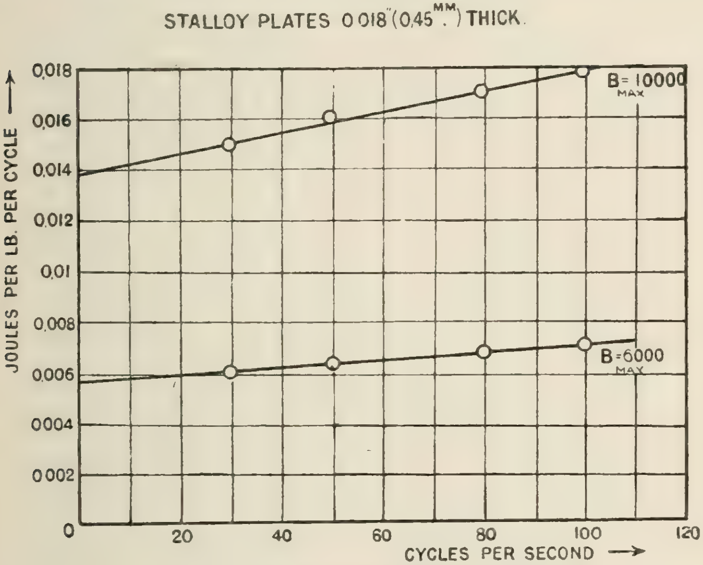
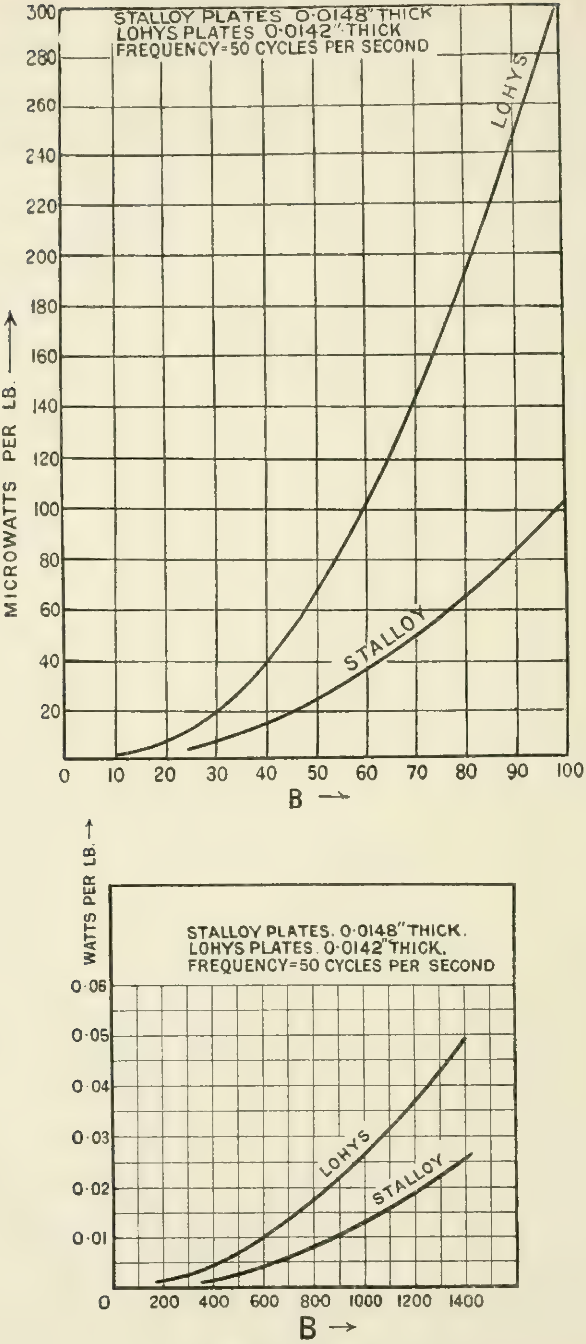


FIG. 418.

(§ 115), and it is clear from these curves that the iron losses for stalloy are very much less than for ordinary transformer iron.

Messrs. Lysaght also manufacture special low loss alloyed iron, and in Fig. 421 the losses of this alloy are shown, as well as the losses of good quality armature iron also supplied by Messrs. Lysaght.

275. Effect of Eddy Currents on the Distribution of the Magnetic Induction in Iron Plates—Skin Effect.—It was pointed out in § 273 (Fig. 414) that the eddy currents due to an alternating flux through an iron plate result in a weakening of the alternating magnetic flux. A little consideration will show that this weakening action will be a maximum in the central part of the section of the plate, since the sum of the eddy currents


STALLOY PLATES 0·0148" THICK
LOHYS PLATES 0·0142" THICK
FREQUENCY=50 CYCLES PER SECOND
MICROWATTS PER LB. ⟶
300
280
260
240
220
200
180
160
140
120
100
80
60
40
20
0
10
20
30
40
50
60
70
80
90
100
B ⟶
LOHYS
STALLOY
WATTS PER LB. ⟶
STALLOY PLATES. 0·0148" THICK.
LOHYS PLATES. 0·0142" THICK.
FREQUENCY=50 CYCLES PER SECOND
0·06
0·05
0·04
0·03
0·02
0·01
0
200
400
600
800
1000
1200
1400
B ⟶
LOHYS
STALLOY

Figs. 419, 420.

is effective in weakening the flux there. This concentration of the flux towards the outer or skin layers of the iron is known as the *skin effect,* and is of the same nature as that considered in §§ 269 and 272. This effect has been mathematically investigated by Sir J. J. Thomson.

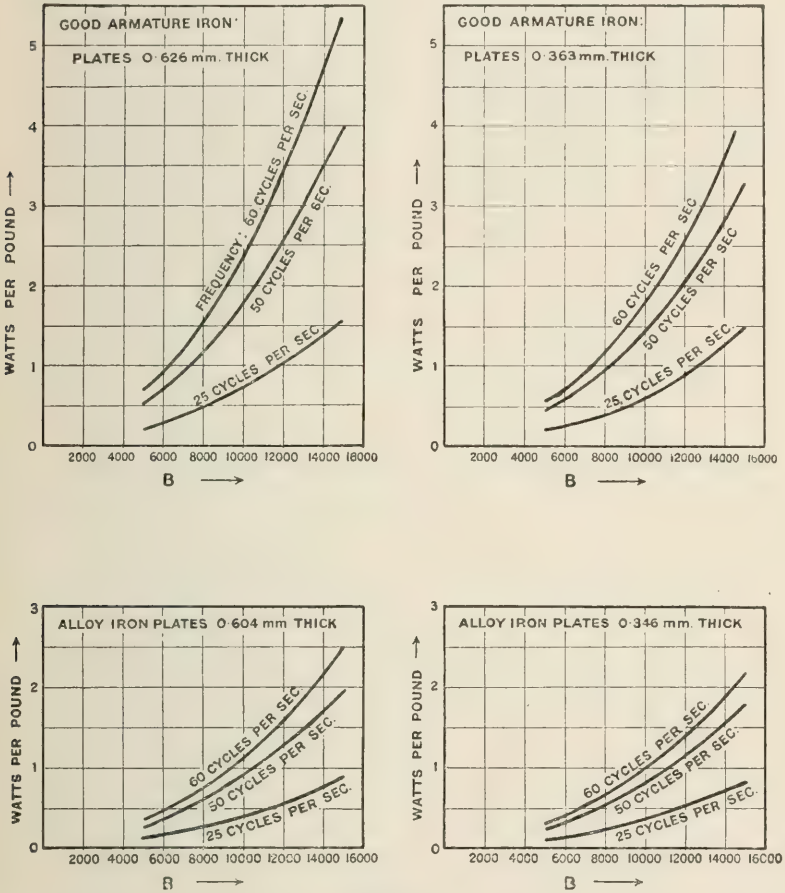


Fig. 421.

In Fig. 422 the curves show the distribution of the flux density across the breadth of the plate, the total breadth being 2Δ. If B_{max} represents the maximum value of the alternating flux density at the surface of the plate, and if B_x is the maximum value of the alternating flux density at distance x from the centre of the plate (see Fig. 416), the ratio $\frac{B_x}{B_{max}}$ is plotted in Fig. 422 for plates 0·25 mm., 1 mm., and 2 mm. thick respectively, the frequency of the alternating flux being 50 cycles p. s., the permeability $\mu = 2{,}000$, and the conductivity $\lambda = 10^5$.

It will be seen that even for plates 0·75 mm. thick there is an appreciable weakening of the flux density in the central part of the plate, whilst for plates 3 mms. thick the flux is almost wiped out in the central part of the plate.

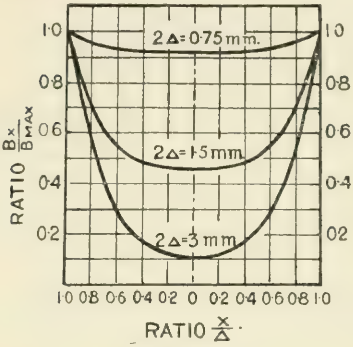


FIG. 422.

This *skin effect* is more marked at higher values of the frequency.

C. P. Steinmetz has treated this problem in detail, and gives the following results: If σ is the penetration in cms. —*i.e.*, the thickness of the surface layer which at constant induction would give the same total flux as actually exists in the whole section of the iron—then σ is proportional to $\frac{1}{\sqrt{\lambda \mu f}}$, where λ is the electrical conductivity of the material, μ is the magnetic permeability, and f is the frequency.[1]

The values of σ in cms. for soft iron and cast iron respectively are shown in the following table for various values of the frequencies:

Frequency.	25	50	500	10^5
Soft iron $\mu = 2{,}000 : \lambda = 10^5$ ..	0·0501	0·036	0·0113	0·0008
Cast steel $\mu = 1{,}000 : \lambda = 5 \times 10^4$ ..	0·101	0·071	0·023	0·0016

It is apparent, therefore, that in alternating current apparatus, such as electric solenoids, it is essential that the whole magnetic circuit should be finely laminated for an efficient performance to be obtained.

276. Power Transformer.—A transformer is an apparatus which is used to transform electrical energy from a voltage V_1 to a voltage V_2. It consists essentially of two coils which are arranged in relationship of mutual induction to each other. For low frequencies the mutual magnetic circuit of the two coils is generally of laminated iron. For very high frequencies, however, such as are common in wireless telegraphy, iron is not used on account of the very high losses which would take place in the iron (see § 273) and also the magnitude of the *skin effect* (§ 275) becomes prohibitive.

Electrical energy is supplied to one coil, the primary, and the other coil, the secondary, is then used to supply the consumer.

Thus, suppose energy is transmitted from a power station at a pressure V_1 volts, and by means of the transformer is transformed down to a pressure of V_2 volts. If the consumer then draws a current of I_2 ampère from the transformer secondary a current I_1 will flow in the primary, such that

$$V_1 I_1 = V_2 I_2$$

very approximately.

Suppose in Fig. 423 two separate coils A and B are wound on a ring of iron, and let coil A have w_1 turns and coil B have w_2 turns.

Suppose, first, that coil B is on open circuit and coil A connected to a source of sinusoidal alternating current p. d. of V_1 volts r. m. s. value.

An alternating magnetic flux will be established in the iron ring which will almost entirely be confined to the iron. For the present this flux will be considered as being entirely confined to the iron, and will therefore link all the turns of each of the coils A and B.

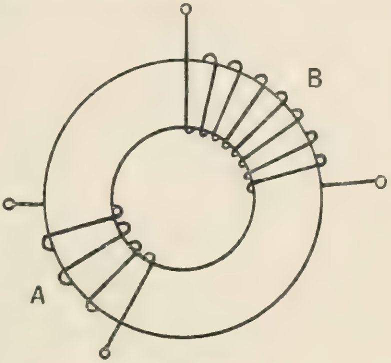


FIG. 423.

Let the supply frequency be f cycles per second, and let Φ be the *maximum* value of the flux through the coil during the cycle.

The magnitude of the *back e. m. f.* induced in the primary coil at any instant will be (§ 197)—

$$e_1 = \frac{w_1}{10^8}\frac{d\Phi}{dt} \text{ volts.}$$

Now the flux changes from $+\Phi$ to $-\Phi$ during the time of half a cycle—*i.e.*, in $\frac{1}{2f}$ second.

Hence the mean rate of change of flux during a cycle is $2\Phi \times 2f = 4\Phi f$, and the mean value of the induced e. m. f. in the primary coil will be—

$$E_{mean} = \frac{w_1}{10^8} 4\Phi f \text{ volts.}$$

It was shown on p. 339 that for sinusoidal waves the ratio of the r. m. s. value to the mean value is 1·11. Hence the r. m. s. value of the induced back e. m. f. in the primary coil is—

$$E_1 = \frac{4{\cdot}44\Phi w_1 f}{10^8} \text{ volts} = V_1 \text{ very approximately.}$$

Since the flux Φ is assumed to link all the turns of the secondary winding, the r. m. s. value of the e. m. f. induced in the secondary winding will be—

$$E_2=\frac{4{\cdot}44\Phi w_2 f}{10^8}\text{ volts,}$$

so that

$$\frac{E_1}{E_2}=\frac{w_1}{w_2},$$

or the ratio of transformation is the ratio of the number of turns in the respective windings.

Suppose now that the primary winding A is connected to a source of alternating current supply, so that the p. d. at the terminals of the winding A is $\overline{V}\cos\omega t$ volts.

Let L_1 henrys be the coefficient of self-induction of the primary winding A.

Let L_2 henrys be the coefficient of self-induction of the secondary winding B, including the inductance of the consumer's load connected to B.

Let M henrys be the coefficient of mutual induction of the two windings A and B.

Let R_1 ohms be the resistance of the primary winding A.

Let R_2 ohms be the resistance of the secondary winding B, including the resistance of the consumer's load connected to B.

Let $\omega=2\pi\times$frequency of supply.

Then the following simultaneous equations hold—viz.:

$$\overline{V}\cos\omega t=L_1\frac{di_1}{dt}+M\frac{di_2}{dt}+R_1 i_1,$$

$$o=L_2\frac{di_2}{dt}+M\frac{di_1}{dt}+R_2 i_2,$$

where i_1 and i_2 represent the current in ampères in the windings A and B respectively.

The solution of these equations is—

$$i_1=\overline{I}_1\cos(\omega t-\phi_1),$$

$$i_2=\overline{I}_2\cos(\omega t-\phi_2),$$

where

$$\overline{I}_2{}^2=\frac{M^2\omega^2}{L_2{}^2\omega^2+R_2}\overline{I}_1{}^2,$$

$$\overline{I}_1{}^2=\frac{\overline{V}^2}{L^2\omega^2+R^2},$$

$$L=L_1-\frac{M^2L_2\omega^2}{L^2\omega^2+R_2{}^2},$$

$$R=R_1+\frac{M^2R_2\omega^2}{L_2{}^2\omega^2+R_2{}^2},$$

$$\tan\phi_1=\frac{L\omega}{R}:\tan(\phi_2-\phi_1)=-\frac{R_2}{L_2\omega}.$$

The simplest method of obtaining this solution is by the use of complex quantities (see Chapter XXII., Example III.).

It is thus seen that the equivalent coefficient of self-induction L of the primary winding A, when the secondary winding B is loaded, is *less than* the coefficient of self-induction L_1 of the primary A, when the secondary B is open. Similarly the equivalent resistance R of the primary winding A, when the secondary winding B is loaded, is *greater than* the true resistance R_1 of the primary A. That is to say, mutual induction has the effect of diminishing the effective inductance of a coil and increasing the effective resistance.

For many problems a graphical solution is much more easily dealt with than the above algebraical solution, and for this purpose the conception of the *equivalent mutual flux* has been developed, and is briefly explained in the following:

Consider two coils I and II having w_1 and w_2 turns respectively. Let L_1, L_2 henrys be the respective coefficients of self-induction, and M henrys the coefficient of mutual induction.

Let a current of i_1 ampères flow in coil I. The e. m. f. induced in coil II will be—

$$e_2 = M \frac{di_1}{dt} \text{ volts.}$$

This may be written $e_2 = \Phi w_2 \frac{di_1}{dt}$, where Φ is an equivalent mutual flux supposed to link all the w_2 turns of coil II.

The e. m. f. induced in coil I due to the mutual induction flux is—

$$e_1^1 = M \frac{di_1}{dt} \frac{w_1}{w_2} = \Phi w_1 \frac{di_1}{dt} \text{ volts.}$$

The back e. m. f. induced in coil I due to the total flux linkages is—

$$e_1 = L_1 \frac{di_1}{dt}.$$

The e. m. f.

$$L_1 \frac{di_1}{dt} - M \frac{w_1}{w_2} \frac{di_1}{dt}$$

$$= \left(L_1 - M \frac{w_1}{w_2}\right) \frac{di_1}{dt}$$

$$= S_1 \frac{di_1}{dt} \text{ volts,}$$

is the e. m. f. due to leakage flux, and S_1 may be termed the co-efficient of leakage induction of coil I.

Similarly the e. m. f.—

$$\left(L_2 - M\frac{w_2}{w_1}\right)\frac{di_2}{dt}$$
$$=S_2\frac{di_2}{dt}$$

is the e.m.f. induced in coil II due to leakage flux, and S_2 may be termed the coefficient of leakage induction of coil II.

It should be noted that the flux to which S_2 is due is not necessarily pure leakage, since the flux to which it is due may link *some* of the turns of coil I, and, *vice versa*, the flux to which S_1 is due may link *some* of the turns of coil II.

Suppose $S_1=0$ and $S_2=0$—that is to say, there is no flux leakage in either coil.

Then $$L_1=M\frac{w_1}{w_2}: L_2=M\frac{w_2}{w_1},$$

or $$L_1L_2=M^2;$$

that is— $$M=\sqrt{L_1L_2},$$

and this is the greatest value which M can have.

The expression $$\frac{L_1}{M\frac{w_1}{w_2}}=\frac{L_1}{L_1-S_1}=\sigma$$

is termed the Hopkinson leakage coefficient.

277. Vector Diagram of the Transformer.—In the first place, it is a great simplification to assume that the primary and secondary winding have each the same number of turns. Suppose for the primary—

S_1 henrys is the coefficient of leakage self-induction as defined in § 276.

R_1 ohms is the resistance of the winding.

w_1 is the number of turns in the winding.

Let $S_2 : R_2 : w_2$ be the corresponding quantities for the secondary winding.

If it be assumed that the number of turns in the secondary winding is made equal to the number in the primary—*i.e.*, the secondary turns increased in the ratio $\frac{w_1}{w_2}$, then in order that this assumed number of turns may be electrically identical with the actual number of turns, it is necessary to assume (1) that the secondary current is *reduced* in the ratio $\frac{w_2}{w_1}$ so that the number of ampère turns of the secondary may be unchanged, (2) that the resistance R_2 is *increased* in the ratio $\left(\frac{w_1}{w_2}\right)^2$ so that the i^2R loss in the secondary may be unchanged, and (3) that the coefficient S_2 is *increased* in the ratio $\left(\frac{w_1}{w_2}\right)^2$ so that the pressure drop and power factor of the secondary may be unchanged.

Thus, if $S_2^1 : L_2^1 : w_2^1$ are the constants of the equivalent secondary winding—

$$w_2^1 = w_2\left(\frac{w_1}{w_2}\right) = w_1 : i_2^1 = \left(\frac{w_2}{w_1}\right) i_2,$$

$$R_2^1 = R_2\left(\frac{w_1}{w_2}\right)^2 : (i_2^1)^2 R_2^1 = i_2^2 R_2,$$

$$S_2^1 = S_2\left(\frac{w_1}{w_2}\right)^2.$$

It is further assumed that the mutual flux remains constant throughout the variations of load. Strictly, the more practical case is that in which the applied p. d. at the primary terminals is kept constant. The error introduced by the assumption of constant mutual flux is, however, small, and a very great simplification obtained by making this assumption of constant mutual flux. The equations deduced in § 276 are, however, exact if the coefficients L_1, L_2, etc., are constant—*i.e.*, if the iron is not saturated.

In Fig. 424 let Φ be the mutual flux and let OI_0 be the current in the primary which is necessary to produce this flux. If iron losses are neglected the current OI_0 will be in phase with the flux Φ. The current OI_0 is the magnetising current of the transformer, all the vectors representing r.m.s. values.

The flux Φ will induce a back e. m. f. OE_1 in the primary and a back e. m. f. OE_2 in the secondary. The e. m. f. OE_2 will produce a current in the secondary circuit, such that if L_2 and r_2 are the inductance and resistance respectively of the secondary load, then

$$I_2 = \frac{E_2}{\sqrt{[(L_2+S_2^1)\omega]^2+(R_2^1+r_2)^2}} \text{ and } \tan\phi_{o2} = \frac{(L_2+S_2^1)\omega}{R_2^1+r_2}$$

and thus the vector OI_2 may be drawn. The primary current OI_1 must be the vector sum of the magnetising current OI_0 and the secondary current OI_2, and so the vector OI_1 may be drawn.

The p. d. applied to the primary terminals must be sufficient to overcome the back e. m. f. OE_1, the leakage reactance volts $S_1\omega I_1$, and the resistance volts R_1I_1. Thus the vector of applied p. d. OV_1 is the sum of the vectors OE_1, R_1I_1, and ωS_1I_1. The angle of lag of the primary current on the applied p. d. is ϕ_1.

The terminal pressure of the secondary will be given by OV_2, and V_2E_2 is the vector of pressure drop in the secondary, of which $R_2^1I_2^1$ is the resistance drop and $\omega S_2^1I_2^1$ the leakage reactance drop.

The assumption that the mutual flux Φ is constant means that the vector OE_1 is assumed to be constant, whereas actually the vector OV_1—*i.e.*, the applied p. d.—should be taken as constant. Since, however, in a good transformer the quantities S_1 and R_1 are usually very small, the vectors OV_1 and OE_1 will be nearly equal and coincident.

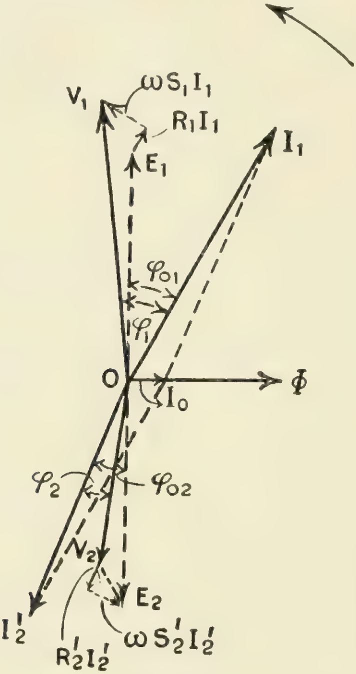


FIG. 424.

By assuming various loads on the secondary of the transformer the p. d. at the terminals of the secondary may be determined, and the angle ϕ_1 of the phase displacement of the primary current on the primary p. d.

The percentage drop, which is an important factor of a transformer, is defined by the ratio—

$$100 \, \frac{\text{secondary p. d. at open circuit, secondary p. d. on load}}{\text{secondary p. d. on open circuit}}.$$

For example, this ratio should be of the order of 2 per cent. for a well-designed 10 kilowatt transformer on full non-inductive load.

If the effect of iron losses is to be considered in the vector diagram the vector of magnetising current OI_o will be in advance of the flux vector $O\Phi$, the phase difference being such that the component of OI_o' in the direction of OE_1 will be $i_o{}^1$, where $Ei_o{}^1$ =iron losses.

CHAPTER XXI

TRANSIENT ELECTRIC CURRENTS

278. A Direct Current P. D. applied to a Circuit containing Resistance, Self-Induction, and Capacity in Series.—In Fig. 425 is shown the series circuit connected to a secondary battery through a switch. At the moment of closing the switch a current will flow in the circuit for a very short time. Eventually the current dies away, leaving the condenser charged to the p. d. of the secondary battery. Such a current is termed a **transient current,** and the study of transient currents is becoming one of increasing practical importance.

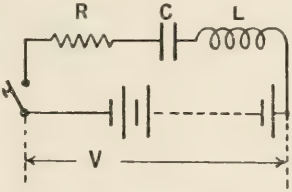


FIG. 425.

For the circuit shown in Fig. 425 the following equation holds:

$$L\frac{di}{dt}+Ri+\frac{1}{C}\int idt=V,$$

in which C farads is the capacity of the condenser, $\int idt=Q$ coulombs is the quantity with which the condenser is charged at any instant, and V volts is the p. d. applied to the terminals of the series connection.

Differentiating the above equation with respect to the time gives—

$$\frac{d^2i}{dt^2}+\frac{R}{L}\frac{di}{dt}+\frac{i}{CL}=0.$$

The solution of this equation depends upon the relative magnitudes of the constants R, L, and C, and three cases are to be distinguished—viz.:

Case I. $\frac{1}{4}\frac{R^2}{L^2}$ is less than $\frac{1}{CL}$.

Case II. $\frac{1}{4}\frac{R^2}{L^2}=\frac{1}{CL}$.

Case III. $\frac{1}{4}\frac{R^2}{L^2}$ is greater than $\frac{1}{CL}$.

Case I.—$\frac{1}{4}\frac{R^2}{L^2}$ *is less than* $\frac{1}{CL}$.

The solution of the equation is then of the form—

$$i = \mathrm{A}e^{-\frac{1}{2}\frac{\mathrm{R}}{\mathrm{L}}t} \cos\left[\sqrt{\frac{1}{\mathrm{CL}} - \frac{1}{4}\frac{\mathrm{R}^2}{\mathrm{L}^2}}\, t + \mathrm{B}\right],$$

where A and B are constants, the values of which depend upon the initial conditions. If the initial conditions are such that

$$t = 0 \ : \ i = 0 \ : \int i\,dt = 0,$$

this denotes that at the instant of closing the switch the condenser is uncharged. By substituting these values for t and i in the equation for i, it follows that either $\mathrm{B} = \frac{\pi}{2}$ or $\mathrm{A} = 0$. But A cannot be zero, as otherwise the current i would always be zero, hence

$$\mathrm{B} = \frac{\pi}{2}.$$

By substituting the values $i = 0$ and $\int i\,dt = 0$ in the original differential equation it follows that—

$$\left.\frac{di}{dt}\right]_{t=0} = \frac{\mathrm{V}}{\mathrm{L}}.$$

By differentiating the equation for i and substituting $t = 0 : i = 0 : \left.\frac{di}{dt}\right]_{t=0} = \frac{\mathrm{V}}{\mathrm{L}}$, it is found that—

$$\mathrm{A} = -\frac{\mathrm{V}}{\mathrm{L}\sqrt{\frac{1}{\mathrm{CL}} - \frac{1}{4}\frac{\mathrm{R}^2}{\mathrm{L}^2}}}.$$

Hence the equation for the current is—

$$i = \frac{\mathrm{V}}{\mathrm{L}\sqrt{\frac{1}{\mathrm{CL}} - \frac{1}{4}\frac{\mathrm{R}^2}{\mathrm{L}^2}}}\, e^{-\frac{1}{2}\frac{\mathrm{R}}{\mathrm{L}}} \sin\sqrt{\frac{1}{\mathrm{CL}} - \frac{1}{4}\frac{\mathrm{R}^2}{\mathrm{L}^2}}\, t \text{ ampères}$$

and similarly for any other initial conditions the corresponding equation for the current may be obtained.

Example.—As an example suppose $\mathrm{C} = \frac{200}{10^6}$ farads—*i.e.*, 200 micro-farads: $\mathrm{L} = 0{\cdot}005$ henry: $\mathrm{R} = \frac{1}{2}$ ohm: $\mathrm{V} = 200$ volts.

Then $$i = 40e^{-50t} \sin 1000t \text{ ampères},$$

which shows that the current is an oscillating current of gradually decreasing amplitude and of frequency $f = \frac{1000}{2\pi} =$

160 cycles per second. This current wave is plotted in Fig. 426.

Case II.—$\frac{1}{4}\frac{R^2}{L^2}=\frac{1}{CL}$.

The solution of the differential equation for this case is—

$$i=(\mathrm{A}t+\mathrm{B})e^{-\frac{1}{2}\frac{\mathrm{R}}{\mathrm{L}}t},$$

where A and B are constants of which the values depend upon the initial conditions.

Let the initial conditions be—

$$t=0:i=0:\int idt=0$$

as before.

Then $$\mathrm{B}=0 \text{ and } \left.\frac{di}{dt}\right]_{t=0}=\frac{\mathrm{V}}{\mathrm{L}}.$$

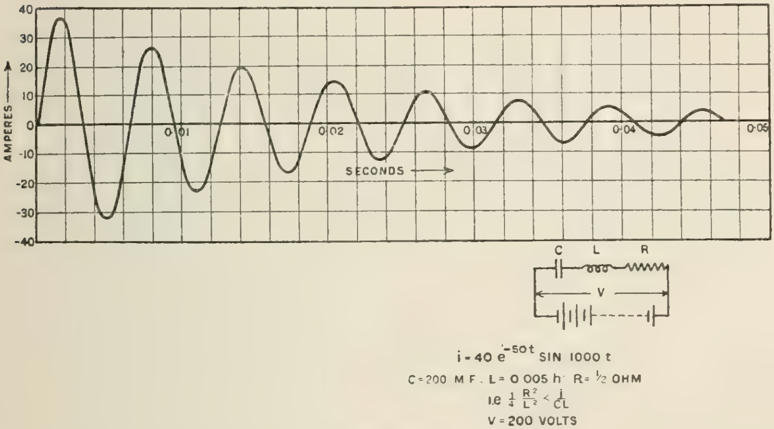


FIG. 426.

Also by differentiating the equation for i—

$$\frac{di}{dt}=-\frac{1}{2}\frac{\mathrm{R}}{\mathrm{L}}e^{-\frac{1}{2}\frac{\mathrm{R}}{\mathrm{L}}t}(\mathrm{A}t)+\mathrm{A}e^{-\frac{1}{2}\frac{\mathrm{R}}{\mathrm{L}}t}$$

and hence when $$t=0:\left.\frac{di}{dt}\right]_{t=o}=\frac{\mathrm{V}}{\mathrm{L}},$$

and $$\mathrm{A}=\frac{\mathrm{V}}{\mathrm{L}},$$

therefore the current in the circuit will be—

$$i=\frac{\mathrm{V}}{\mathrm{L}}te^{-\frac{1}{2}\frac{\mathrm{R}}{\mathrm{L}}t} \text{ ampères.}$$

Example.—As an example, assume

$$L=0{\cdot}005 \text{ henry}; C=\frac{200}{10^6} \text{ farads}: R=10 \text{ ohms}: \text{and } V=200 \text{ volts}.$$

then $$i=40000te^{-1000t} \text{ ampères}.$$

This current is plotted in Fig. 427.

Case III.—$\frac{1}{4}\frac{R^2}{L^2}$ *is greater than* $\frac{1}{CL}$.

The solution of the differential equation is then—

$$i=Ae^{\lambda_1 t}+Be^{\lambda_2 t},$$

in which λ_1 and λ_2 are the two roots of the equation—

$$\lambda^2+\frac{R}{L}\lambda+\frac{1}{CL}=0,$$

and A and B are constants which are determined by the initial conditions.

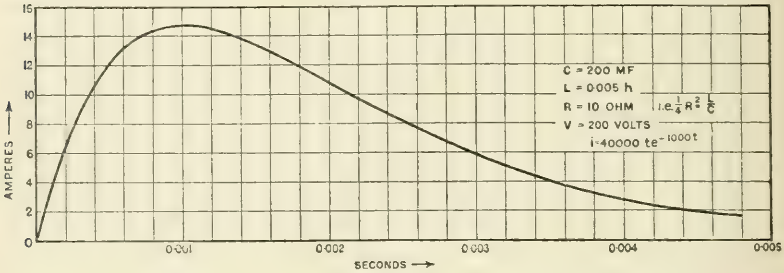


Fig. 427.

If these conditions are $t=0: i=0: \int idt=0$, as before, then $A+B=0$, or $A=-B$.

Further $\left.\frac{di}{dt}\right]_{t=o}=\frac{V}{L}$ for these initial conditions.

By differentiating the equation for i and substituting the value $\left.\frac{di}{dt}\right]_{t=o}=\frac{V}{L}$, it follows that—

$$A=\frac{V}{L}\;\frac{1}{(\lambda_1-\lambda_2)},$$

and the current in the circuit will be—

$$i=\frac{V}{L}\;\frac{1}{(\lambda_1-\lambda_2)}\left(e^{\lambda_1 t}-e^{\lambda_2 t}\right) \text{ ampères}.$$

Example.—As an example, suppose

$$L=0{\cdot}005 \text{ henry}: C=\frac{200}{10^6} \text{ farads}: R=20 \text{ ohms}: V=200 \text{ volts}.$$

Then $\lambda_1 = -3730$, $\lambda_2 = -270$, and $A = -11 \cdot 5$,

therefore $\quad i = 11 \cdot 5\,(e^{-270t} - e^{-3730t})$ ampères.

This current is plotted in Fig. 428.

If in any of the above cases the condenser is not without a charge when the switch is closed (see Fig. 425), the equation for the current is obtained by suitably altering the constants in the corresponding solution of the differential equation to satisfy the given initial conditions. Thus, if the condenser has a charge Q_0 coulombs when the switch is closed, the initial conditions are $t=0 : i=0 : \int i dt = Q_0$, and the corresponding values of the constants can be obtained. Similarly for any other given initial conditions.

If the condenser is removed from the circuit the rise of current in the circuit when the switch is closed will be as shown in Fig. 262, § 204.

In a similar way, the transient current may be deduced for a circuit containing a condenser and resistance and connected to a source of direct current supply.

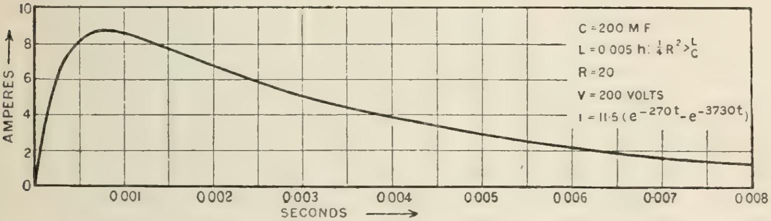


Fig. 428.

The transient current which flows when a condenser discharges through a non-inductive resistance has been dealt with in p. 203, Fig. 172.

The transient current which flows when a condenser discharges through an inductive coil and a resistance in series is given by the solutions of the equation—

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i dt = 0,$$

and is precisely similar in form to the solutions given (p. 440) for the charging current of a condenser when connected through an inductive coil and a resistance to a source of direct current supply.

The initial conditions in this case are $t=0 : i=0 : \int i dt = Q_0$, where Q_0 is the charge of the condenser in coulombs when the discharge commences.

For example, if $\frac{1}{4}\frac{R^2}{L^2}$ is less than $\frac{1}{CL}$, the current will be given by the equation—

$$i=\frac{Q_0}{CL\sqrt{\frac{1}{CL}-\frac{1}{4}\frac{R^2}{L^2}}}e^{-\frac{1}{2}\frac{R}{L}t}\sin\sqrt{\frac{1}{CL}-\frac{1}{4}\frac{R^2}{L^2}}\,t \text{ ampères.}$$

In Fig. 429 are shown two oscillograms giving the current which flows when a condenser discharges through a self-induction and a non-inductive resistance in series. In the oscillogram A the resistance was small compared with the inductance. In B the resistance was much larger than in A, the other conditions being the same. It is to be observed that the oscillations in B die away much more quickly than in A, although the periodic time is approximately the same in both cases. The periodic time is—

$$T=\frac{1}{\text{frequency}}=\frac{2\pi}{\sqrt{\frac{1}{CL}-\frac{1}{4}\frac{R^2}{L^2}}}$$

or for relatively small values of R is very approximately—

$$T=\frac{1}{\text{frequency}}=2\pi\sqrt{CL},$$

C farads being the capacity and L henrys the inductance. Fig. 429 has been supplied by the Cambridge and Paul Instrument Co., Ltd.

If $\frac{1}{4}\frac{R^2}{L^2}$ is equal to or greater than $\frac{1}{CL}$ the discharge of the condenser will be non-oscillatory.

279. Discharge of an Inductive Coil through a Non-inductive Resistance.—The current which flows when a coil of self-induction L henry and carrying a current I ampères is disconnected from the supply mains and connected across a non-inductive resistance (Fig. 430) may be obtained as follows:

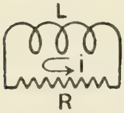


FIG. 430.

If i ampères is the current at any moment the energy stored in the self-induction is $\frac{1}{2}Li^2$ joules (see p. 285). The energy dissipated in the resistance is i^2R joules per second, and this energy is obtained from the energy stored in the magnetic field of the coil of self-induction L henrys.

Hence the rate of loss of energy stored in the inductance is—

$$=i^2R,$$

or

$$-\frac{d}{dt}(\tfrac{1}{2}Li^2)=Ri^2,$$

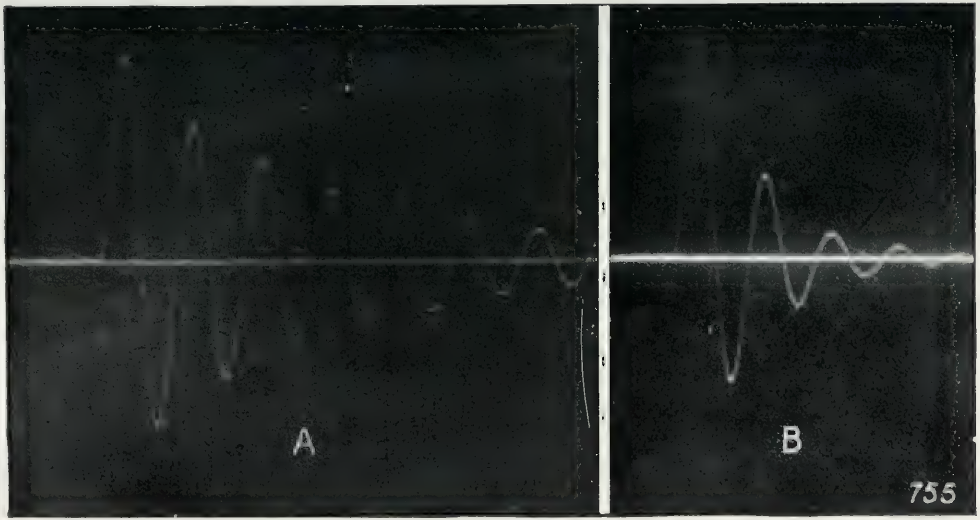


FIG. 429.

To face page 444

or
$$L\frac{di}{dt}=-Ri;$$

that is,
$$\frac{di}{i}=-\frac{R}{L}\,dt.$$

The solution of this equation is—

$$i=Ae^{-\frac{R}{L}t},$$

where A is a constant depending on the conditions at the commencement of the discharge.

At time $t=0$: $i=I$, and after an indefinitely long time $i=0$.

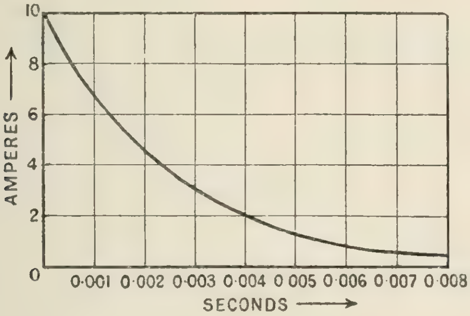


Fig. 431.

Hence
$$i=Ie^{-\frac{R}{L}t}\text{ ampères.}$$

In Fig. 431 this current is shown for the conditions—

I=10 ampères: R=2 ohms: L=0·005 henry.

280. An Alternating Current P. D. applied to a Circuit containing Resistance and Self-induction in Series.—The circuit is shown in Fig. 432, and it is required to find what the current will be immediately after closing the switch—that is, before the steady cyclic conditions of current have been established as determined by the formula given in § 229.

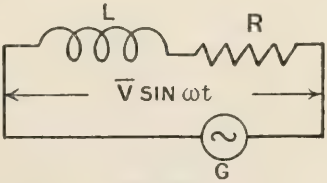


Fig. 432.

At any moment the current in the circuit will be given by the equation—

$$L\frac{di}{dt}+Ri=\overline{V}\sin\omega t,$$

where
$$\omega=2\pi\times\text{frequency}.$$

The solution of this equation is—

$$i = Ae^{-\frac{R}{L}t} + \frac{\bar{V}}{\sqrt{R^2+(\omega L)^2}} \sin(\omega t - \beta),$$

in which A is a constant and $\tan \beta = \frac{\omega L}{R}$ (p. 330). The first term of this equation contains an exponential function of the time, and gives the transient current which flows after the switch is closed. In practice this term rapidly disappears, and the current assumes the steady cyclic state given by the equation—

$$i = \frac{\bar{V}}{\sqrt{R^2+(\omega L)^2}} \sin(\omega t - \beta) \text{ ampères.}$$

If the initial conditions—*i.e.*, at the instant at which the switch is closed—are

$$t = 0 : i = 0 : \bar{V} \sin \omega t = 0,$$

then

$$A = \frac{\bar{V}}{\sqrt{R^2+(\omega L)^2}} \sin \beta.$$

and the transient current is—

$$i_1 = \frac{\bar{V}}{\sqrt{R^2+(\omega L)^2}} e^{-\frac{R}{L}t} \sin \beta \text{ ampères.}$$

Hence

$$i = \frac{\bar{V}}{\sqrt{R^2+(\omega L)^2}} \left[e^{-\frac{R}{L}t} \sin \beta + \sin(\omega t - \beta) \right].$$

Example.—As an example, suppose $R = 2$ ohms: $\omega = 2\pi$ frequency $= 2\pi \times 100 = 628$: $L = 0{\cdot}02$ henry: $\bar{V} = 200$ volts. Then

$$\tan \beta = \frac{\omega L}{R} = 6{\cdot}28 : \beta = 81^\circ = 1{\cdot}41 \text{ radians} : \sin \beta = 0{\cdot}988.$$

The transient term of the current is therefore

$$i_1 = \frac{200}{\sqrt{2^2+(12{\cdot}56)^2}} \, 0{\cdot}988 \, e^{-100t} \text{ ampère;}$$

that is,

$$i_1 = 15{\cdot}5 \, e^{-100t} \text{ ampères.}$$

The second or permanent term of the current is

$$i_2 = 15{\cdot}8 \sin(628\,t - 1{\cdot}41) \text{ ampères.}$$

In Fig. 433 the two terms of the current wave are shown, and in Fig. 434 the resultant current wave is shown. It will be observed that after a time corresponding to about three complete cycles of the supply p. d. (*i.e.*, after about 0·03 second from the time of closing the switch) the transient term has become relatively very small, and the current has reached very nearly a steady cyclic state.

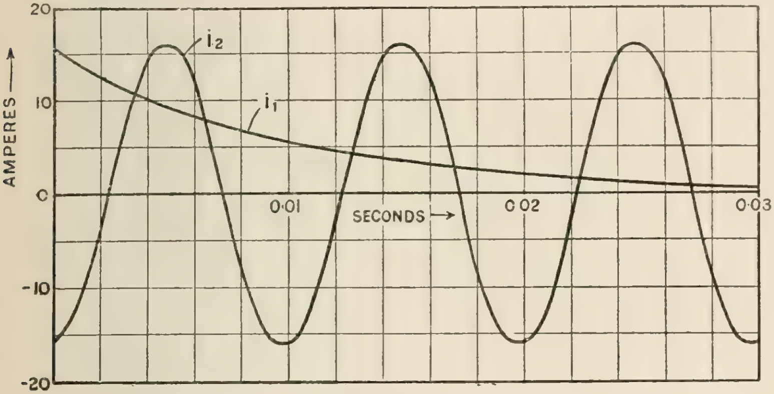


FIG. 433.

As has been stated, the current waves shown in Figs. 433 and 434 correspond to the conditions that the switch is closed at the instant at which the applied p. d. is zero. By suitably altering

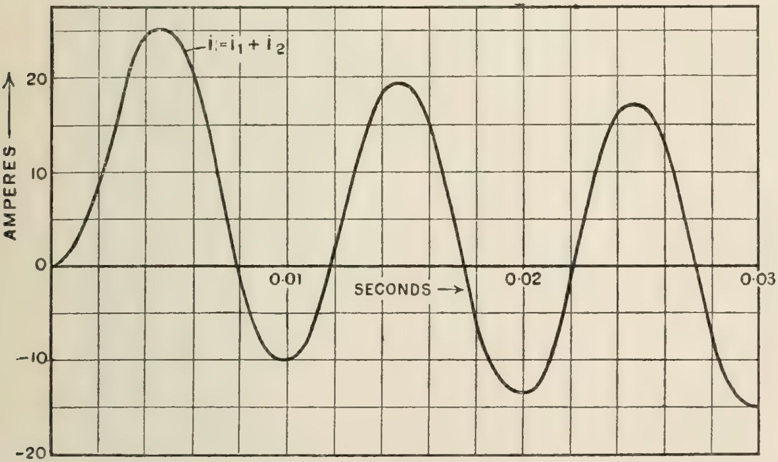


FIG. 434.

the value of the constant A in the foregoing example the corresponding current wave which will result if the switch is closed at any other instant of the p. d. cycle may be deduced.

An interesting special case is that of a circuit of which the resistance R (Fig. 432) is zero, and for which the switch is closed when the applied p.d. is zero. For this case $\tan\beta=\infty$, (*i.e.*, infinity) and $\beta=90°$.

Therefore $$i=\mathrm{A}-\frac{\bar{\mathrm{V}}}{\omega\mathrm{L}}\cos\omega t \text{ ampères.}$$

If the initial conditions are such that

$$t=0:i=0:\bar{\mathrm{V}}\sin\omega t=0,$$

$$\mathrm{A}=\frac{\bar{\mathrm{V}}}{\omega\mathrm{L}},$$

and therefore $$i=\frac{\bar{\mathrm{V}}}{\omega\mathrm{L}}(1-\cos\omega t)\text{ ampères.}$$

This equation shows that the current never becomes negative, the minimum value being $i=0$.

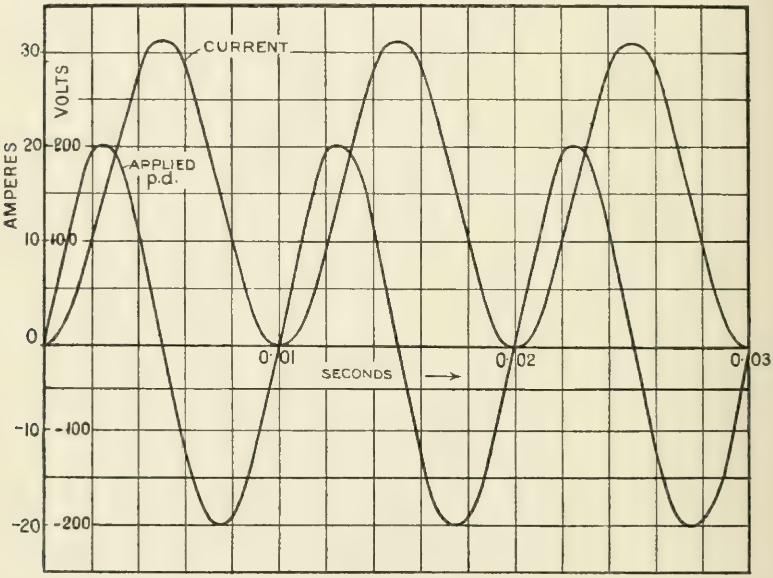


Fig. 435.

Example.—Suppose $\bar{\mathrm{V}}=200$ volts: $\omega=2\pi\times100=628$: $\omega\mathrm{L}=12{\cdot}56$.

$$i=16\,(1-\cos\omega t)\text{ ampères.}$$

This current wave is shown in Fig. 435, the wave of the supply p. d. also being shown. It will be seen that the p. d. and current waves pass through the zero values simultaneously.

281. An Alternating P. D. applied to a Circuit Containing Resistance, Inductance, and Capacity in Series.—The circuit is shown in Fig. 436, and it is assumed that an alternating p. d.

$v=\bar{V}\cos(\omega t-a)$ volts is applied to the terminals. The following equation will then hold:

$$L\frac{di}{dt}+Ri+\frac{1}{C}\int i\,dt=\bar{V}\cos(\omega t-a);$$

that is,
$$L\frac{d^2i}{dt^2}+R\frac{di}{dt}+\frac{i}{C}=\bar{V}\omega\cos\left(\omega t-a+\frac{\pi}{2}\right)\cdot$$

or
$$\frac{d^2i}{dt^2}+\frac{R}{L}\frac{di}{dt}+\frac{i}{LC}=F\cos\left(\omega t-a+\frac{\pi}{2}\right),$$

where
$$F=\frac{\bar{V}\omega}{L}.$$

The solution of this equation is of three forms dependent upon whether $\frac{1}{4}\frac{R^2}{L^2}$ is less than, equal to, or greater than $\frac{1}{LC}$.

Case I.—$\frac{1}{4}\frac{R^2}{L^2}$ *is less than* $\frac{1}{LC}$.

The solution in this case is—

$$i=Ae^{-\frac{1}{2}\frac{R}{L}t}\cos\left\{\sqrt{\frac{1}{LC}-\frac{1}{4}\frac{R^2}{L^2}}\,t+\beta\right\}+Y\cos\left(\omega t-a+\frac{\pi}{2}-\gamma\right)$$

where
$$Y=\frac{F}{\sqrt{\left(\frac{1}{LC}-\omega^2\right)^2+\frac{R^2}{L^2}\omega^2}}$$

and
$$\tan\gamma=\frac{\frac{R\omega}{L}}{\left(\frac{1}{LC}-\omega^2\right)}=\frac{R\omega C}{1-LC\omega^2}$$

and A, β, and a are constants determined by the initial conditions.

If, when the switch is closed, $t=0:i=0:\int i\,dt=0$—that is, the condenser has no charge, and if the applied p. d. is zero at that instant—*i.e.*, $a=\frac{\pi}{2}$, then $\left.\frac{di}{dt}\right]_{t=0}=0$, and the values of the constants A and β may be determined in terms of $\bar{V}$ and the constants L, R, C, and ω.

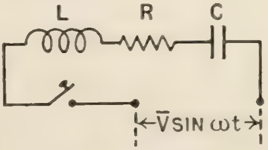


FIG. 436.

Example I.—As an example, suppose $L=0{\cdot}005$ henry: $C=\frac{200}{10^6}$ farad: $R=0{\cdot}5$ ohm, the supply frequency is $f=100$ cycles per second—that is, $\omega=2\pi\times100=628$, and $\bar{V}=200$ volts.

The value of F in the equation for i is—

$$F=\frac{V\omega}{L}=\frac{200\times628}{0\cdot005}=25\cdot12\times10^6.$$

The natural frequency of the circuit is—

$$\frac{1}{2\pi}\sqrt{\frac{1}{LC}-\frac{1}{4}\frac{R^2}{L^2}}=\frac{1}{2\pi}\sqrt{\frac{1}{LC}}\text{ very approximately.}$$

That is, the natural frequency is 160 cycles per second, and $2\pi\times$natural frequency$=2\pi\times160=1000$.

The constant A is therefore equal to 41·5, and the constant $\beta=173\tfrac{1}{2}^\circ=3\cdot03$ radians.

Also $$\tan\gamma=\frac{R\omega C}{1-LC\omega^2}=0\cdot104;$$

that is, $$\gamma=6^\circ=0\cdot105\text{ radians.}$$

Hence

$$i=41\cdot5e^{-50t}\cos(1000t+3\cdot03)+41\cdot3\cos(628t-0\cdot105)\text{ ampères.}$$

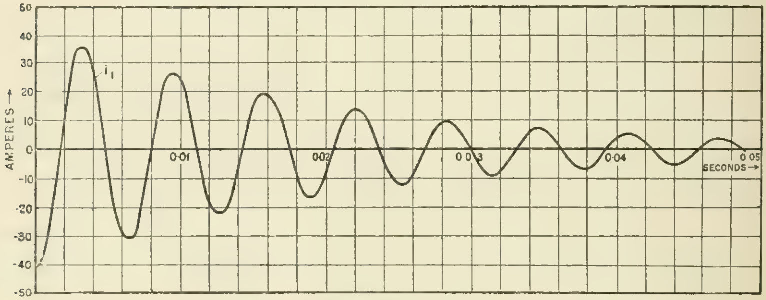


FIG. 437.

The first term is the transient term—viz.:

$$i_1=41\cdot5\,3^{-50t}\cos(1000t+3\cdot03)\text{ ampères,}$$

and this is plotted in Fig. 437, from which it will be seen that after about $\frac{1}{20}$th second after closing the switch the amplitude of this term is less than $\frac{1}{10}$th of its maximum amplitude 41·3. The frequency of this transient current is $\frac{1000}{2\pi}=160$ cycles per second.

The second term is the permanent term—viz.:

$$i_2=41\cdot3\cos(628t-0\cdot105)\text{ ampères,}$$

and this is plotted in Fig. 438. This is the current which flows in the circuit when the conditions have become steady.

The total current in the circuit is—

$$i=i_1+i_2,$$

and this is plotted in Fig. 439. This current eventually becomes identical with the current i_2 (Fig. 438).

Dec. [illegible]

For any other given initial conditions—*e.g.*, if the switch is closed at a moment when the applied p. d. is not zero—the transient current i_1 may be determined by finding the corresponding values of the constants A and β. The permanent term of the current—viz., i_1—is, of course, independent of the initial conditions if the values of L, C, and R are kept constant.

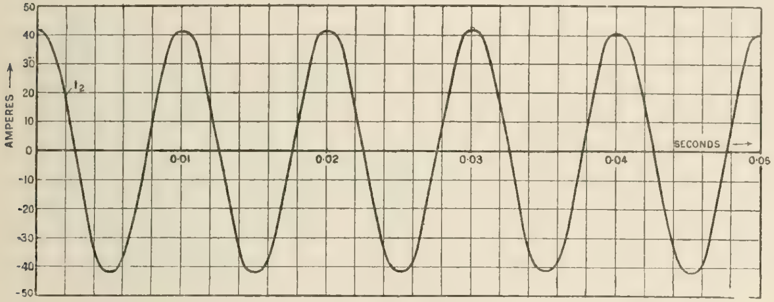


FIG. 438.

Example II.—As another example of the case in which $\frac{1}{4}\frac{R^2}{L^2}$ is less than $\frac{1}{LC}$ will be chosen the condition that the natural frequency of the circuit is the same as the frequency of the applied p. d.—that is, the circuit is in *resonance* with the frequency of the supply pressure.

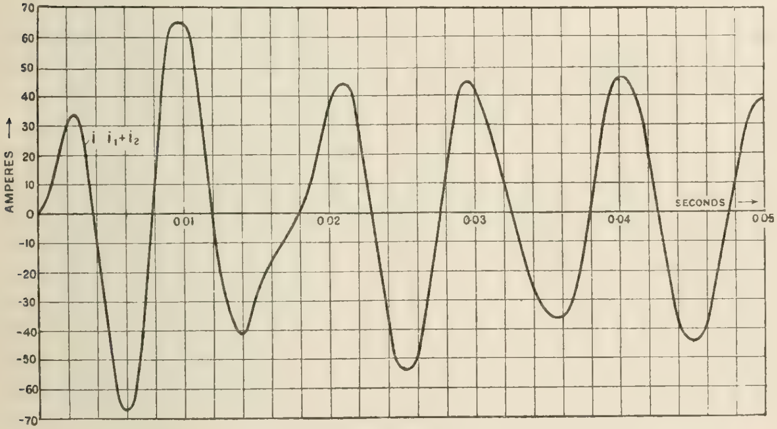


FIG. 439.

Let $L=0{\cdot}005$ henry: $C=\frac{200}{10^6}$ farad: $R=0{\cdot}25$ ohm: $\bar{V}=200$ volts: supply frequency $f=160$ cycles per second: $\omega=2\pi\times160=1000$: $LC\omega^2=1$.

Then in the equation for the current i given on p. 449

$$F=40\times10^6 : \tan\gamma=\infty : \gamma=90^\circ=\frac{\pi}{2} \text{ radians};$$

that is,

$$i=Ae^{-25t}\cos(1000t+\beta)+800\sin 1000t \text{ ampères.}$$

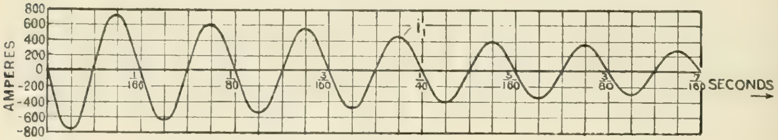


FIG. 440.

If the initial conditions are—

$$t=0 : i=0 : \int i\,dt=0,$$

then $$A=800$$

and $$B=90^\circ=\frac{\pi}{2} \text{ radians.}$$

Hence $$i=-800e^{-25t}\sin 1000t+800\sin 1000t \text{ ampères.}$$

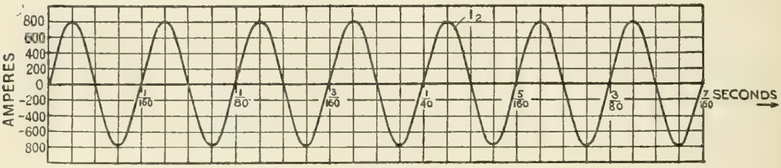


FIG. 441.

The first term of this equation is the transient term—viz.:

$$i_1=-800e^{-25t}\sin 1000t \text{ ampères,}$$

and this is plotted in Fig. 440.

The second term is the permanent term—viz.:

$$i_2=800\sin 1000t \text{ ampères,}$$

and this is plotted in Fig. 441.

The resultant current is—

$$i=i_1+i_2,$$

and this is given in Fig. 442, from which it is seen that the current is of gradually increasing amplitude until the steady condition is reached for which i_1 has become zero and $i=i_2$.

Case II.—Suppose $\frac{1}{4}\frac{R^2}{L^2}$ *is greater than* $\frac{1}{LC}$.

The solution of the differential equation (p. 449) for the current in the series circuit is then—

$$i=Ae^{\lambda_1 t}+Be^{\lambda_2 t}+\frac{F}{\sqrt{\left(\frac{1}{LC}-\omega^2\right)^2+\left(\frac{R\omega}{L}\right)^2}}\cos\left(\omega t-a+\frac{\pi}{2}-\gamma\right),$$

where A, B, and a are constants of which the values depend upon the initial conditions.

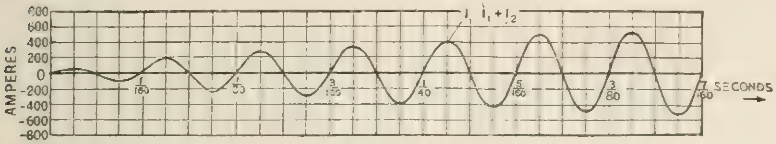


FIG. 442.

As an example, suppose the supply frequency is 50 cycles per second, and let the constants of the circuit be as follows: $R=10{\cdot}1$ ohms; $L=0{\cdot}005$ henry; $C=200\times10^{-6}$ farads; $\overline{V}=200$ volts.

Then $\omega=2\pi\times\text{frequency}=314$; $\tan\gamma=0{\cdot}705$; $\gamma=35\tfrac{1}{4}°=0{\cdot}616$.

$$F=\frac{\overline{V}\omega}{L}=12{\cdot}56\times10^6 : \frac{1}{LC}=10^6:$$

$$\lambda_1=-1{,}151;\ \lambda_2=-868.$$

The current in the circuit is therefore—

$$i=Ae^{-1151t}+Be^{-868t}+11{\cdot}4\cos\left(314t-a+\frac{\pi}{2}-0{\cdot}616\right),$$

where A, B, and a are constants depending upon the initial conditions.

Suppose when the circuit switch is closed the applied p. d. is at its maximum positive value—that is, $a=o$ (see p. 449).

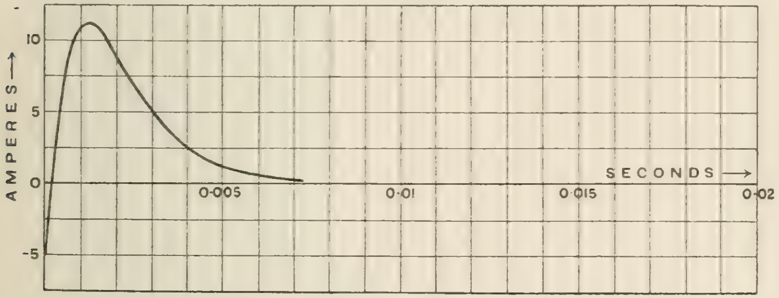


FIG. 443.

Further, let the current be zero and the condenser without charge when the switch is closed. That is, the initial conditions are—

$$t=o;\ i=o;\ \int idt=o;\ a=o;\ \frac{di}{dt}=40{,}000.$$

By substitution in the equations for i and $\frac{di}{dt}$ it is found that the constants have the following values:

$$A=-131{\cdot}6;\ B=125.$$

Hence the current is—

$$i = -131{\cdot}6e^{-1151t} + 125e^{-868t} + 11{\cdot}4 \cos(314t + 0{\cdot}954) \text{ ampères.}$$

The transient component of this current is:

$$i_1 = -131{\cdot}6e^{-1151t} + 125e^{-868t} \text{ ampères,}$$

and this is plotted in Fig. 443.

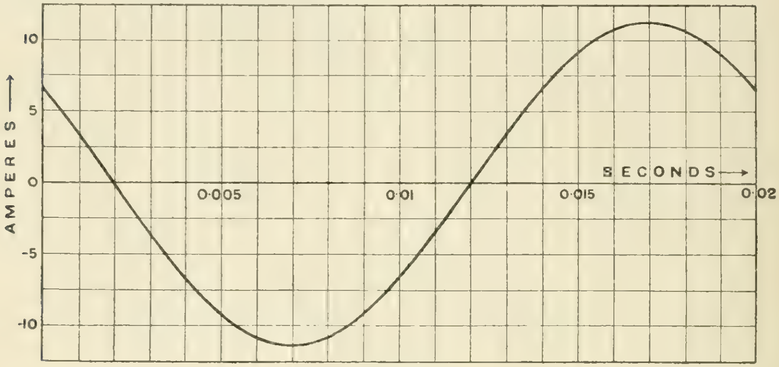


FIG. 444.

The permanent component of the current is—

$$i_2 = 11{\cdot}4 \cos(314t + 0{\cdot}954) \text{ ampères,}$$

and this is plotted in Fig. 444.

The actual resultant current is—

$$i = i_1 + i_2,$$

and this is shown in Fig. 445.

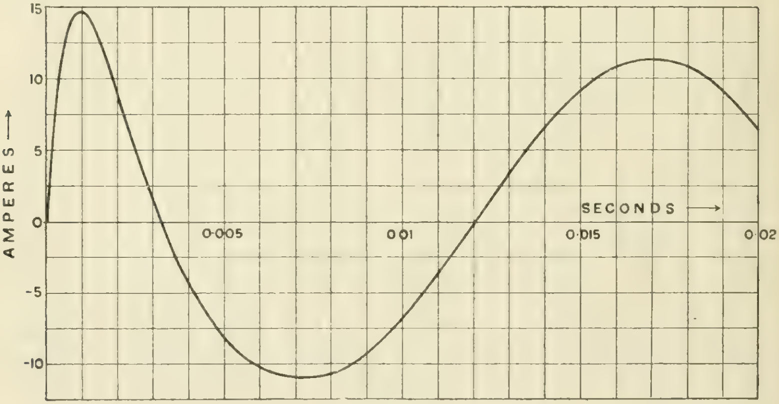


FIG. 445.

The case in which $\frac{1}{4}\frac{R^2}{L^2}$ is equal to $\frac{1}{LC}$ may be dealt with in a similar manner, by making use of the solution given on p. 441.

The calculation of the transient currents may be greatly facilitated by the use of the table given in the Appendix.

CHAPTER XXII

USE OF COMPLEX QUANTITIES IN ALTERNATING CURRENT PROBLEMS

282. Definitions of some Mathematical Functions.—It is known from the Binomial Theorem that—

$$\left(1+\frac{1}{n}\right)^n=1+1+\frac{1-\frac{1}{n}}{\underline{|2}}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\underline{|3}}+\ . \ . \ .$$

where $\underline{|2}=1\times 2$: $\underline{|3}=1\times 2\times 3$, and so on.

Hence, if n is indefinitely large ($n=\infty$),

$$\left(1+\frac{1}{\infty}\right)^\infty=1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\ . \ . \ .,$$

and this is the definition of the base of natural logarithms usually denoted by e.

That is,
$$e=1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\ . \ . \ .$$
$$=2\cdot 718.$$

Similarly,

$$\left[\left(1+\frac{1}{n}\right)^n\right]^x=1+x+\frac{x^2\left(1-\frac{1}{nx}\right)}{\underline{|2}}+\frac{x^3\left(1-\frac{1}{nx}\right)\left(1-\frac{2}{nx}\right)}{\underline{|3}}+\ . \ . \ .;$$

that is, if n is indefinitely large—

$$e^x=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ . \ . \ .,$$

also
$$e^{-x}=1-x+\frac{x^2}{\underline{|2}}-\frac{x^3}{\underline{|3}}+\ . \ . \ .$$

The quantities e^x and e^{-x} are called exponential functions of x.

If j is $\sqrt{-1}$,

$$e^{jx}=1+jx-\frac{x^2}{\underline{|2}}-j\frac{x^3}{\underline{|3}}+\frac{x^4}{\underline{|4}}\ . \ . \ .$$
$$e^{-jx}=1-jx-\frac{x^2}{\underline{|2}}+j\frac{x^3}{\underline{|3}}+\frac{x^4}{\underline{|4}}+\ . \ . \ .$$

It is proved in mathematical textbooks that—

$$\sin x = x - \frac{x^3}{\underline{|2}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \dots$$

$$\cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \dots$$

Hence
$$\cos x = \tfrac{1}{2}\left(e^{jx} + e^{-jx}\right)$$

$$j \sin x = \tfrac{1}{2}\left(e^{jx} - e^{-jx}\right).$$

Also
$$\cos x + j \sin x = e^{jx}$$

$$\cos x - j \sin x = e^{-jx}.$$

Certain other combinations of the exponential functions have properties closely analogous in form to the ordinary trigonometrical functions. These functions are called the hyperbolic sine, hyperbolic cosine, hyperbolic tangent, etc., and they bear in some respects the same relation to the rectangular hyperbola that the circular (trigonometrical) functions bear to the circle. These hyperbolic functions are:

$$\cosh x = \tfrac{1}{2}\left(e^{x} + e^{-x}\right) = 1 + \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} + \dots$$

$$\sinh x = \tfrac{1}{2}\left(e^{x} - e^{-x}\right) = x + \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} + \dots$$

$$\tanh x = \frac{\sinh x}{\cosh x} : \ \cosh x = \frac{\cosh x}{\sinh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} : \ \operatorname{cosech} x = \frac{1}{\sinh x}.$$

Further, it will be easily seen from the formulæ already given that—

$$\cosh jx = \cos x : \ \sinh jx = j \sin x$$

$$\tanh jx = j \tan x$$

$$\cosh jx + \sinh jx = e^{jx}.$$

283. Representation of Vectors by Complex Quantities.—In Fig. 446 let OA be a line drawn in the direction OX and of length a units.

That is—

$$OA = a.$$

A line OA^1 drawn in the direction OX^1 and of length a units is written—

$$OA^1 = -a;$$

that is,
$$OA^1 = -OA.$$

Now assume that a line OB (Fig. 446) drawn in the direction OY and of length a units be written—

$$OB = j(OA) = ja,$$

the prefix j denoting that OB is drawn at right angles to the direction of OA, the angle being measured in a counter-clockwise direction from OA.

According to this notation $j(OB)$ is a line drawn at right angles to OB, measuring the angle in a counter-clockwise direction from OB.

That is— $$j(OB) = OA^1,$$

or $$j(ja) = OA^1 = -a;$$

hence $$j^2 a = -a$$

$$j^2 = -1,$$

and $$j = \sqrt{-1}.$$

Hence j must be assumed to be $\sqrt{-1}$ in order that the prefix j satisfies the condition that $j(OA)$ denotes a line drawn at right angles to OA in the counter-clockwise direction, and that—

$$OA^1 = -OA.$$

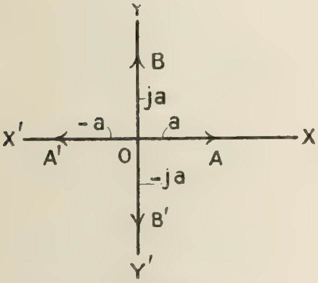


Fig. 446.

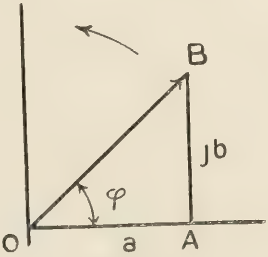


Fig. 447.

In Fig. 447 let a vector OB be drawn at an angle ϕ to the axis OX, and let the length of OB be r units. This vector may be resolved into two components at right angles—viz., OA along OX, and AB in the direction OY.

Let OA be a units and AB be b units, so that $OA = a$ and $AB = jb$.

The vector OB may then be written—

$$OB = a + jb.$$

Similarly a vector OB^1 (Fig. 448) in the second quadrant is—

$$OB^1 = -a + jb.$$

A vector in the third quadrant is—

$$OB^{11} = -a - jb,$$

and a vector in the fourth quadrant is—

$$OB^{111} = a - jb.$$

Since the length of the vector OB (Fig. 447) is r units—

$$a = r \cos \phi : \ b = r \sin \phi$$

$$r = \sqrt{a^2 + b^2} : \ \tan \phi = \frac{b}{a},$$

and

$$OB = a + jb = r (\cos \phi + j \sin \phi),$$

and since it has been shown that j under this interpretation must be $\sqrt{-1}$ it follows from the previous results that—

$$OB = r (\cos \phi + j \sin \phi) = re^{j\phi}.$$

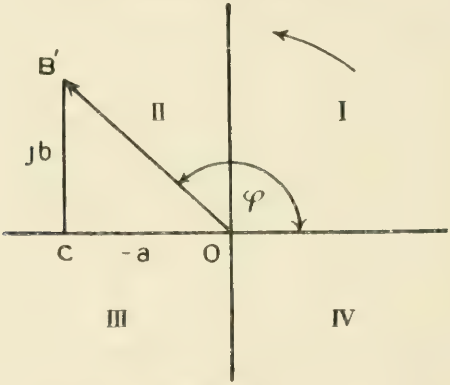


FIG. 448.

Quantities of the form $a + jb$ and the equivalent forms $r (\cos \phi + j \sin \phi)$, and $re^{j\phi}$ are termed **complex quantities.**

The quantity r is the magnitude or the *modulus* of the vector, and ϕ is the *argument*, or $\tan \phi$ *the slope* of the vector.

In Fig. 449 let OA represent a vector in the first quadrant, so that—

$$OA = a + jb;$$

then

$$\begin{aligned} j(OA) &= j(a + jb) \\ &= ja - b \\ &= OA_1. \end{aligned}$$

Hence, if a vector OA is multiplied by j a vector OA_1 is obtained which is of the same magnitude as OA, but makes an angle of 90° with OA, the angle being measured in the counter-clockwise direction—*that is to say, multiplication of a vector by j gives a vector of equal magnitude, but* 90° *ahead of the original vector.*

Similarly, multiplication of a vector by $-j$ gives a vector of equal magnitude, but 90° *behind* the original vector.

Addition of Vectors.—Suppose now that two vectors OA and OB (Fig. 450) defined respectively by—

$$\mathrm{OA}=a+jb=r_1e^{j\phi_1},$$

and $$\mathrm{OB}=c+jd=r_2e^{j\phi_2}$$

are to be added.

Then $$\mathrm{OA}+\mathrm{OB}=a+c+j\ (b+d);$$

that is to say, the resultant vector is obtained by adding the real and imaginary parts respectively.

The slope of the resultant vector is—

$$\tan\phi=\frac{b+d}{a+c},$$

and the magnitude is—

$$\sqrt{(a+c)^2+(b+d)^2}.$$

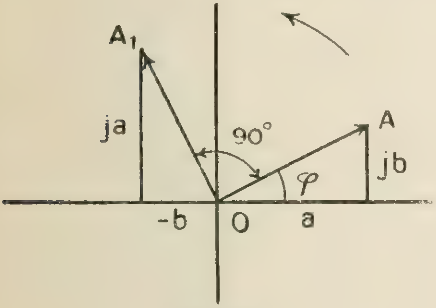


FIG. 449.

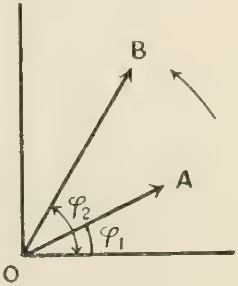


FIG. 450.

Subtraction of Vectors.—

$$\mathrm{OA}-\mathrm{OB}=a-c+j\ (b-d)$$

$$\tan\phi=\frac{b-d}{a-c},$$

and the magnitude is—

$$\sqrt{(a-c)^2+(b-d)^2}.$$

Multiplication of Vectors.—For multiplication the vectors are most conveniently expressed in the form

$$\mathrm{OA}=r_1e^{j\phi_1}$$

and $$\mathrm{OB}=r_2e^{j\phi_2};$$

then $$\mathrm{OA}\times\mathrm{OB}=r_1r_2e^{j(\phi_1+\phi_2)}.$$

Hence the magnitude of the resultant vector is the product of the magnitudes of the respective components, and the slope of the resultant is the sum of the slopes of the components. If the vectors be written in the form—

$$\mathrm{OA}=r_1\,(\cos\phi_1+j\sin\phi_1):\ \mathrm{OB}=r_2\,(\cos\phi_2+j\sin\phi_2);$$

and $$\mathrm{OA}\times\mathrm{OB}=r_1r_2\,[(\cos(\phi_1+\phi_2)+j\sin(\phi_1+\phi_2)],$$

as is also obvious from the previous result.

Division of Vectors.—

$$\frac{\mathrm{OA}}{\mathrm{OB}}=\frac{r_1}{r_2}\,e^{(j\phi_1-\phi_2)}$$

and hence,

$$\frac{\mathrm{OA}}{\mathrm{OB}}=\frac{r_1\,(\cos\phi_1+j\sin\phi_1)}{r_2\,(\cos\phi_2+j\sin\phi_2)}=\frac{r_1}{r_2}\,[\cos(\phi_1-\phi_2)+j\sin(\phi_1-\phi_2)].$$

To Raise a Vector to the nth Power.—

$$(\mathrm{OA})^n=(r_1e^{j\phi_1})^n=r_1{}^ne^{j\phi_1n},$$

and $$r^n\,(\cos\phi_1+j\sin\phi_1)^n=r^n\,(\cos n\phi_1+j\sin n\phi_1).$$

To Determine the nth Root of a Vector.—

$$(\mathrm{OA})^{\frac{1}{n}}=(r_1e^{j\phi_1})^{\frac{1}{n}}=r_1{}^{\frac{1}{n}}e^{j\phi_1\frac{1}{n}},$$

and $$r_1{}^{\frac{1}{n}}\,(\cos\phi_1+j\sin\phi_1)^{\frac{1}{n}}=r_1{}^{\frac{1}{n}}\left(\cos\frac{\phi_1}{n}+j\sin\frac{\phi_1}{n}\right).$$

If, as the result of such operations, an expression of the form $\frac{l+m}{h+jk}$ is obtained, this expression may be reduced to the standard form $a+jb$ by multiplying by $\frac{h-jk}{h-jk}$.

Thus $$\frac{l+m}{h+jk}=\frac{(l+m)\,(h-jk)}{h^2+k^2}=\frac{(l+m)h-jk\,(l+m)}{h^2+k^2};$$

or $$\frac{l+m}{h+jk}=\frac{(l+m)h}{h^2+k^2}-j\,\frac{k(l+m)}{h^2+k^2},$$

and this represents a vector OA in Fig. 451,

where $$\mathrm{O}d=\frac{(l+m)h}{h^2+k^2},$$

and $$d\mathrm{A}=\frac{k(l+m)}{h^2+k^2}.$$

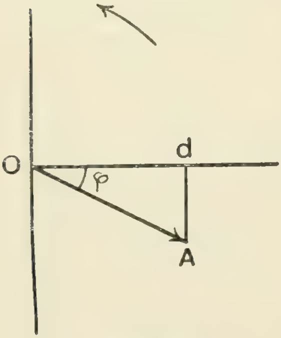


Fig. 451.

284. Example I. A Series Circuit.—Consider a circuit (Fig. 452) containing a resistance R ohms and a reactance X ohms in series, where $\mathrm{X}=\omega\mathrm{L}-\frac{1}{\omega\mathrm{C}}$, as in p. 330, Chapter XVII.

Let the current be given by the vector OI in Fig. 453, and, at the instant considered assumed to be inclined at the angle θ to the horizontal. This vector may be represented for algebraical purposes by the symbol $\underset{\cdot}{\mathrm{I}}$, so that—

$$\underset{\cdot}{\mathrm{I}}=\bar{\mathrm{I}}\,(\cos\theta+j\sin\theta),$$

$\bar{\mathrm{I}}$ being the amplitude of the current vector (see p. 323).

The vector of applied p. d.—viz., OV in Fig. 453—may be represented by the symbol $\underset{\cdot}{\mathrm{V}}$.

Then $\underset{\cdot}{\mathrm{V}}=\underset{\cdot}{\mathrm{I}}\mathrm{R}+\underset{\cdot}{\mathrm{I}}j\mathrm{X}$ (see also §§ 226 and 227, Chapter XVII.)

$$=\underset{\cdot}{\mathrm{I}}\,(\mathrm{R}+j\mathrm{X})=\underset{\cdot}{\mathrm{I}}\left(\mathrm{R}+j\left\{\omega\mathrm{L}-\frac{1}{\omega\mathrm{C}}\right\}\right).$$

In Fig. 453 oa is $\underset{\cdot}{\mathrm{I}}\mathrm{R}$ and ab is $\underset{\cdot}{\mathrm{I}}j\mathrm{X}$, so that ab is at right angles to oa. The vector OV is thus in advance of the vector OI by the angle ϕ where $\tan\phi=\frac{\mathrm{X}}{\mathrm{R}}$.

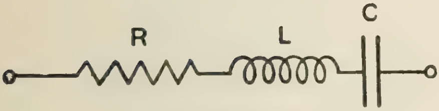


FIG. 452.

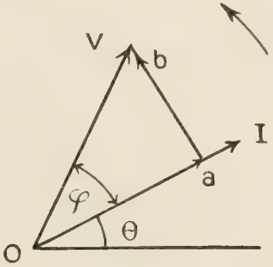


FIG. 453.

The expression $\mathrm{R}+j\mathrm{X}$ denotes the vector of impedance $\underset{\cdot}{\mathrm{Z}}$ of the circuit, so that—

$$\underset{\cdot}{\mathrm{Z}}=\mathrm{R}+j\mathrm{X}=\mathrm{Z}\,(\cos\phi+j\sin\phi),$$

where $$\mathrm{Z}=\sqrt{\mathrm{R}^2+\mathrm{X}^2}\text{ and }\tan\phi=\frac{\mathrm{X}}{\mathrm{R}}.$$

In Fig. 454 OZ is the vector of impedance, and hence

$$\underset{\cdot}{\mathrm{V}}=\underset{\cdot}{\mathrm{I}}\underset{\cdot}{\mathrm{Z}}=\bar{\mathrm{I}}\,(\cos\theta+j\sin\theta)\,\mathrm{Z}\,(\cos\phi+j\sin\phi)$$

$$=\bar{\mathrm{I}}\mathrm{Z}\{\cos(\theta+\phi)+j\sin(\theta+\phi)\}.$$

The vector OV is thus inclined at an angle ϕ in advance of the current vector OI, and $\tan\phi=\frac{\mathrm{X}}{\mathrm{R}}$, which is the same result as given in Fig. 453.

If the circuit has no capacity in series, then $\mathrm{X}=\omega\mathrm{L}$ and $\tan\phi=\frac{\omega\mathrm{L}}{\mathrm{R}}$.

If the circuit has capacity in series but no inductance, then $X = -\frac{1}{\omega C}$ and $\tan\phi = -\frac{1}{\omega CR}$, the current being now *in advance* of the applied p. d.

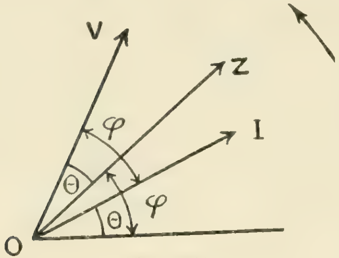


FIG. 454.

Usually it is more convenient in dealing with series circuits to draw the current vector horizontally and refer the position of all other vectors to the position of the current vector. In this case $\theta = 0$, and the vector of applied p. d. will coincide in position with the impedance vector.

285. Admittance, Conductance, Susceptance—Example II. Circuits in Parallel.—In a series circuit of impedance Z, the current vector is given by—

$$\underset{\cdot}{I} = \frac{\underset{\cdot}{V}}{\underset{\cdot}{Z}} = \frac{\underset{\cdot}{V}}{R + jX} = \underset{\cdot}{V}\,\frac{R - jX}{R^2 + X^2}$$

$$= \underset{\cdot}{V}\left[\frac{R}{R^2 + X^2} - j\,\frac{X}{R^2 + X^2}\right]$$

$$= \underset{\cdot}{V}\,[g - jb].$$

That is, $\underset{\cdot}{I} = \underset{\cdot}{V}\underset{\cdot}{Y}$,

where $\underset{\cdot}{Y} = g - jb = \frac{1}{\underset{\cdot}{Z}}$,

and is called the *admittance* of the circuit. The quantity $g = \frac{R}{R^2 + X^2}$ is called the *conductance*, and the quantity $b = \frac{X}{R^2 + X^2}$ is called the *susceptance*.

The magnitude of the admittance is $Y = \sqrt{g^2 + b^2}$. The quantities Y, g, and b are measured in "reciprocal ohms" or *mhos*.

Suppose two circuits are arranged in parallel as in Fig. 455.

Then $\underset{\cdot}{I}_1 = \underset{\cdot}{V}\underset{\cdot}{Y}_1$

$\underset{\cdot}{I}_2 = \underset{\cdot}{V}\underset{\cdot}{Y}_2$

and $\underset{\cdot}{I} = \underset{\cdot}{I}_1 + \underset{\cdot}{I}_2 = \underset{\cdot}{V}(\underset{\cdot}{Y}_1 + \underset{\cdot}{Y}_2)$,

or
$$\underset{\cdot}{I} = \underset{\cdot}{V}\,(g_1 - jb_1 + g_2 - jb_2)$$
$$= \underset{\cdot}{V}\,[g_1 + g_2 - j\,(b_1 + b_2)]$$
$$= \underset{\cdot}{V}\left[\frac{R_1}{R_1{}^2 + X_1{}^2} + \frac{R_2}{R_2{}^2 + X_2{}^2} - j\left(\frac{X_1}{R_1{}^2 + X_1{}^2} + \frac{X_2}{R_2{}^2 + X_2{}^2}\right)\right].$$

The magnitude of the current $\underset{\cdot}{I}$ is

$$\overline{I} = \overline{V}\sqrt{\left(\frac{R_1}{R_1{}^2 + X_1{}^2} + \frac{R_2}{R_2{}^2 + X_2{}^2}\right)^2 + \left(\frac{X_1}{R_1{}^2 + X_1{}^2} + \frac{X_2}{R_2{}^2 + X_2{}^2}\right)^2}.$$

The resultant current in the mains across which two circuits are connected in parallel is thus found in an exceedingly simple manner. The simplification of the problem by the use of complex quantities will be at once seen by comparing the solution of the same problem by the algebraical method given on p. 336, Chapter XVII.

If it be required to find the equivalent resistance and the equivalent reactance of the two parallel circuits the above equation connecting $\underset{\cdot}{I}$ and $\underset{\cdot}{V}$ may be written—

$$\underset{\cdot}{V}=\frac{\underset{\cdot}{I}}{\left[\frac{R_1}{R_1^2+X_1^2}+\frac{R_2}{R_2^2+X_2^2}-j\left(\frac{X_1}{R_1^2+X_1^2}+\frac{X_2}{R_2^2+X_2^2}\right)\right]},$$

$$\text{or}\quad \underset{\cdot}{V}=\underset{\cdot}{I}\,\frac{\frac{R_1}{R_1^2+X_1^2}+\frac{R_2}{R_2^2+X_2^2}+j\left(\frac{X_1}{R_1^2+X_1^2}+\frac{X_2}{R_2^2+X_2^2}\right)}{\left(\frac{R_1}{R_1^2+X_1^2}+\frac{R_2}{R_2^2+X_2^2}\right)^2+\left(\frac{X_1}{R_1^2+X_1^2}+\frac{X_2}{R_2^2+X_2^2}\right)^2};$$

that is, $$V=I\,(R+jX),$$

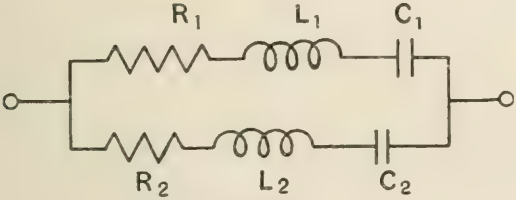


FIG. 455.

where R is the equivalent resistance and X is the equivalent reactance.

$$\text{Hence}\quad R=\frac{\frac{R_1}{R_1^2+X_1^2}+\frac{R_2}{R_2^2+X_2^2}}{\left(\frac{R_1}{R_1^2+X_1^2}+\frac{R_2}{R_2^2+X_2^2}\right)^2+\left(\frac{X_1}{R_1^2+X_1^2}+\frac{X_2}{R_2^2+X_2^2}\right)^2},$$

$$\text{and}\quad X=\frac{\frac{X_1}{R_1^2+X_1^2}+\frac{X_2}{R_2^2+X_2^2}}{\left(\frac{R_1}{R_1^2+X_1^2}+\frac{R_2}{R_2^2+X_2^2}\right)^2+\left(\frac{X_1}{R_1^2+X_1^2}+\frac{X_2}{R_2^2+X_2^2}\right)^2}.$$

286. Example III. A Transformer.—The diagrammatical representation of a transformer is shown in Fig. 456 (see also § 276, Chapter XX.). The current in the primary circuit is given by—

$$V=I_1(R_1+jX_1)+jI_2X.$$

For the secondary circuit—

$$0=\underset{\cdot}{I}_2\,(R_2+jX_2)+j\underset{\cdot}{I}_1X,$$

where $$X_1=\omega L_1:\ X_2=\omega L_2:\ X=\omega M.$$

Hence

$$\underset{\cdot}{V}=\underset{\cdot}{I}_1\,(R_1+jX_1)+jX\left[\frac{-j\underset{\cdot}{I}_1X}{R_2+jX_2}\right]$$

$$=\underset{\cdot}{I}_1\left[R_1+jX_1+\frac{X^2}{R_2+jX_2}\right]$$

$$=\underset{\cdot}{I}_1\left[\frac{R_1R_2+jX_1R_2+jR_1X_2-X_1X_2+X^2}{R_2+jX_2}\right]$$

$$=\underset{\cdot}{I}_1\left[\frac{R_1R_2-X_1X_2+X^2+j(X_1R_2+R_1X_2)}{R_2^2+X_2^2}\right](R_2-jX_2)$$

$$=\underset{\cdot}{I}_1\left[\frac{R_1R_2^2+X^2R_2+R_1X_2^2+j(X_1R_2^2+X_1X_2^2-X_2X^2)}{R_2^2+X_2^2}\right]$$

$$=\underset{\cdot}{I}_1\left[R_1+\frac{R_2X^2}{R_2^2+X_2^2}+j\left\{X_1-\frac{X_2X^2}{R_2^2+X_2^2}\right\}\right]$$

or $$\underset{\cdot}{V}=\underset{\cdot}{I}_1\,(R_e+jX_e),$$

where R_e is the equivalent resistance and X_e the equivalent reactance which, arranged in series and connected to supply terminals of p. d. $\bar{V}$, would take the same current as actually flows in the primary winding of the transformer.

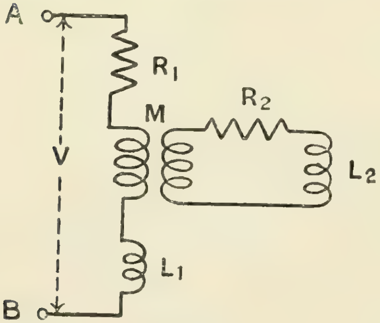


Fig. 456.

Hence the equivalent resistance of the primary circuit is *increased* by an amount $\frac{R_2X^2}{R_2^2+X_2^2}$ when the secondary circuit is closed, and the equivalent inductance of the primary circuit is *decreased* by an amount $\frac{X_2X^2}{R_2^2+X_2^2}$ when the secondary circuit is closed. *In other words, the equivalent resistance of a circuit is increased by mutual induction effects and the equivalent reactance decreased.*

For the current in the transformer secondary—

$$\underset{\cdot}{I}_2=\frac{-jX}{R_2+jX_2}\,\underset{\cdot}{I}_1=\frac{-X(X_2+jR_2)}{R_2^2+X_2^2}\,\underset{\cdot}{I}_1.$$

That is, the amplitude of the secondary current vector is—

$$I_2 = \frac{X}{\sqrt{R_2^2+X_2^2}}\,\bar{I}_1.$$

The vector diagram is shown in Fig. 457, in which—

$$\tan\phi = \frac{X_e}{R_e}$$

$$\tan\theta = \frac{X_2}{R_2}.$$

The above is the mathematical solution of the transformer problem, and all the characteristics of the transformer may be deduced from this solution under the assumption that X_1, X_2, and X are true constants—that is to say, are independent of the current.

As has been previously pointed out, the simple vector diagram given in § 277, Chapter XX., assumes that the value of the mutual flux is constant for all loads, but this is not strictly correct, since the more practical case is the condition that the p. d. applied to the terminals of the primary is constant.

In the above mathematical treatment no such assumption is made. It is to be observed that the value of R_1 must include an amount corresponding to the iron losses in the core of the transformer.

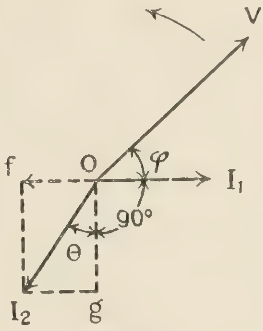


FIG. 457.

287. Example IV. Bridge Method of measuring Capacity and Inductance.—The arrangement of the bridge for this purpose (due to Campbell) is shown in Fig. 458, in which M represents a standard variable mutual inductance, L the inductance, and C the capacity of which the values are to be determined.

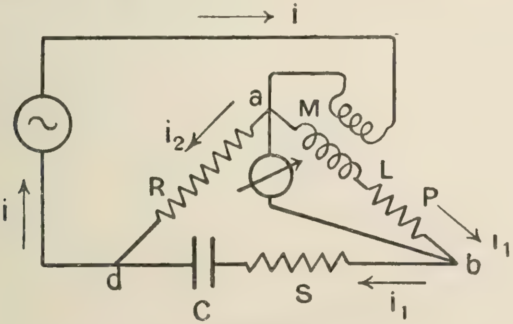


FIG. 458.

The two branch circuits *ad* and *abd* are connected to a source of alternating current supply (preferably of a frequency from 250 to 500 cycles per second), and across the points *ab* a telephone (or alternating current galvanometer) is connected. Adjustment of M and the resistance S is made so that no current flows in the telephone connected across *ab*—that is, the telephone becomes silent when the bridge is balanced, and the p. d. across *ab* accordingly is then zero.

Let the reactance ωL be denoted by x_L.

,, ,, $\frac{1}{\omega C}$,, ,, x_C.

,, ,, ωM ,, ,, x_M.

In the branch *ad* there is only a resistance R, and hence the p. d. across *ad* is $\underset{\cdot}{I}_2 R$.

In the branch *abd* the resistance is P + S and the reactance is $x_L - x_C = \omega L - \frac{1}{\omega C}$.

When balance is obtained the mutual reactance in *ab* will be $x_M = -\omega M$.

Hence the p. d. across *ac* is—

$$\underset{\cdot}{I}_1[(P+S)+j(x_L - x_C)] - jx_M\underset{\cdot}{I}.$$

But this p. d. has already been shown to be—

$$I_2R.$$

Hence $$I_2R = \underset{\cdot}{I}_1[(P+S)+j(x_L - x_C)] - jx_M\underset{\cdot}{I},$$

or $$\underset{\cdot}{I}_2(R+jx_M) = \underset{\cdot}{I}_1[P+S+j(x_L - x_C - x_M)],$$

since $$\underset{\cdot}{I} = \underset{\cdot}{I}_1 + \underset{\cdot}{I}_2.$$

Hence $$\frac{\underset{\cdot}{I}_1}{\underset{\cdot}{I}_2} = \frac{R+jx_M}{[P+S+j(x_L - x_C - x_M)]}.$$

Further, since for a balance the p. d. across *ab* is zero—

$$I_1(P+jx_L) - \underset{\cdot}{I}jx_M = 0,$$

or $$I_1[P+j(x_L - x_M)] = \underset{\cdot}{I}_2 jx_M$$

that is— $$\frac{\underset{\cdot}{I}_1}{\underset{\cdot}{I}_2} = \frac{jx_M}{P+j(x_L - x_M)}$$

$$\therefore \frac{R+jx_M}{[P+S+j(x_L - x_C - x_M)]} = \frac{jx_M}{P+j(x_L - x_M)},$$

which reduces to—

$$PR - x_M x_C = j[Sx_M - R(x_L - x_M)].$$

The real parts of the two sides of this equation must be equal, and also the imaginary parts of the two sides must be equal.

Hence $$\mathrm{PR} = x_{\mathrm{M}} x_{\mathrm{C}},$$

and $$\mathrm{S}x_{\mathrm{M}} = \mathrm{R}(x_{\mathrm{L}} - x_{\mathrm{M}});$$

therefore $$x_{\mathrm{L}} = \frac{\mathrm{S}x_{\mathrm{M}} + \mathrm{R}x_{\mathrm{M}}}{\mathrm{R}},$$

or $$x_{\mathrm{L}} = \frac{x_{\mathrm{M}}(\mathrm{S}+\mathrm{R})}{\mathrm{R}}.$$

That is, $$\mathrm{L} = \mathrm{M}\,\frac{\mathrm{S}+\mathrm{R}}{\mathrm{R}}.$$

Similarly from the equation—

$$\mathrm{PR} = x_{\mathrm{M}} x_{\mathrm{C}}$$

$$\frac{1}{\omega\mathrm{C}} = \frac{\mathrm{PR}}{\omega\mathrm{M}},$$

or $$\mathrm{C} = \frac{\mathrm{M}}{\mathrm{PR}}.$$

When the bridge is balanced, therefore, the following two equations must be satisfied—viz.:

$$\mathrm{L} = \mathrm{M}\,\frac{\mathrm{S}+\mathrm{R}}{\mathrm{R}} : \mathrm{C} = \frac{\mathrm{M}}{\mathrm{PR}}.$$

288. Example V. The Anderson-Fleming Method of measuring Inductance.—The connections for this method are shown in Fig. 459. The inductance L henrys of which the value is to be determined is connected in the arm *cb*, a known capacity is connected across *ad*, and adjustments of the resistances in the arms are made until the p. d. across *dc* is zero—that is, until a telephone connected across these points becomes silent.

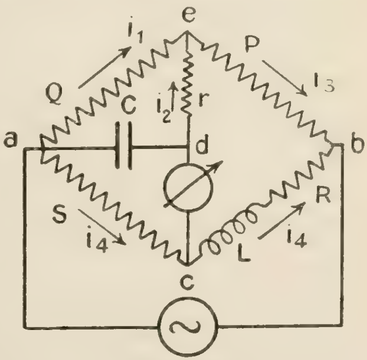


FIG. 459.

Let $\underset{\cdot}{\mathrm{V}}$ be the vector of the supply p. d. connected across *ab*. Then if the resistance in *ae* is Q ohms, in *eb* is P ohms, in *ac* is S ohms, and in *cb* is R ohms, the following equations will hold when balance is obtained:

$$\underset{\cdot}{\mathrm{V}} = \underset{\cdot}{\mathrm{I}}_4\mathrm{S} + \underset{\cdot}{\mathrm{I}}_4\mathrm{R} + \underset{\cdot}{\mathrm{I}}_4\, j\omega\mathrm{L}.$$

$$\mathrm{V} = \underset{\cdot}{\mathrm{I}}_1\mathrm{Q} + \underset{\cdot}{\mathrm{I}}_3\mathrm{P}.$$

$$Q\underset{\cdot}{I}_1=\underset{\cdot}{I}_2r-\underset{\cdot}{I}_2j\frac{1}{\omega C}.$$

$$\underset{\cdot}{I}_3=\underset{\cdot}{I}_1+\underset{\cdot}{I}_2.$$

$$\underset{\cdot}{I}_4S=-\underset{\cdot}{I}_2j\frac{1}{\omega C}.$$

Hence $$\underset{\cdot}{I}_1Q=\underset{\cdot}{I}_2r+\underset{\cdot}{I}_4S,$$

and $$\begin{aligned}V&=\underset{\cdot}{I}_1(P+Q)+\underset{\cdot}{I}_2P\\&=(\underset{\cdot}{I}_2r+\underset{\cdot}{I}_4S)\left(\frac{P+Q}{Q}\right)+\underset{\cdot}{I}_2P\\&=\left[\frac{P+Q}{Q}r+P\right]\left[-\frac{\underset{\cdot}{I}_4S\omega C}{j}\right]+\underset{\cdot}{I}_4S\left(\frac{P+Q}{Q}\right).\end{aligned}$$

Therefore $$S+R+j\omega L=\left[\frac{P+Q}{Q}r+P\right]\left[\frac{-S\omega C}{j}\right]+S\left(\frac{P+Q}{Q}\right),$$

or $$\left[\frac{P+Q}{Q}r+P\right][-S\omega C]+jS\frac{P}{Q}=jR-\omega L.$$

Hence, equating the real parts of both sides of the equation gives—

$$L\omega=S\omega C\left[\frac{P+Q}{Q}r+P\right],$$

or $$L=SC\left[\frac{P}{Q}r+r+P\right],$$

which gives the value of the inductance L henrys in terms of the known capacity C farads and the resistances S, Q, P, and r ohms.

Also by equating the imaginary parts of both sides of the equation—

$$S\frac{P}{Q}=R,$$

or $$\frac{R}{S}=\frac{P}{Q}.$$

This last equation gives the value of the total resistance R ohms in branch *cb*, and shows that a condition of balance across *cd* with an alternating supply across *ab* is that there should be zero p. d. across *ec* when direct current pressure is applied across *ab*, as in the ordinary Wheatstone Bridge measurement (§ 151).

289. Example VI. The Transformation of a Mesh-Connected Circuit to a Star-Connected Circuit.—Let the mesh-connected circuit be as shown in Fig. 460, the impedances of the respective branches being $Z_{\text{I}}:\ Z_{\text{II}}:Z_{\text{III}}$.

Let the impedances of the respective arms of the equivalent star-connected circuit (Fig. 461) be $Z_a : Z_b : Z_c$.

The two arrangements will be identical in so far as the effect on the mains is concerned if the *admittance* between any two mains is the same for both arrangements.

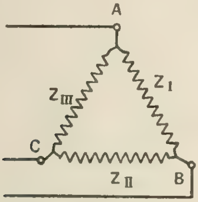


FIG. 460.

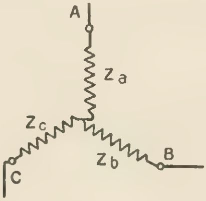


FIG. 461.

The admittance between A and C in Fig. 460 is the admittance of Z_{I} and Z_{II} in series, and this series combination is parallel with Z_{III}.

The admittance of the branch containing Z_{III} is—

$$\underset{.}{Y}_{III}=\frac{1}{\underset{.}{Z}_{III}}.$$

The admittance of the branch containing Z_{I} and Z_{II} in series is—

$$\underset{.}{Y}_{I:\,II}=\frac{1}{\underset{.}{Z}_{I}+\underset{.}{Z}_{II}}$$

The admittance of the two branches in parallel is (see p. 462)—

$$\underset{.}{Y}_{III}+\underset{.}{Y}_{I:\,II}=\frac{1}{\underset{.}{Z}_{III}}+\frac{1}{\underset{.}{Z}_{I}+\underset{.}{Z}_{II}}.$$

The admittance between the lines A and C in Fig. 461 is—

$$\underset{.}{Y}_{a:\,c}=\frac{1}{\underset{.}{Z}_a+\underset{.}{Z}_c}.$$

Hence, if the two systems are identical—

$$\frac{1}{\underset{.}{Z}_{III}}+\frac{1}{\underset{.}{Z}_{I}+\underset{.}{Z}_{II}}=\frac{1}{\underset{.}{Z}_a+\underset{.}{Z}_c}.$$

Similarly

$$\frac{1}{\underset{.}{Z}_{II}}+\frac{1}{\underset{.}{Z}_{I}+\underset{.}{Z}_{III}}=\frac{1}{\underset{.}{Z}_c+\underset{.}{Z}_b}$$

$$\frac{1}{\underset{.}{Z}_{I}}+\frac{1}{\underset{.}{Z}_{II}+\underset{.}{Z}_{III}}=\frac{1}{\underset{.}{Z}_a+\underset{.}{Z}_b}.$$

From which it follows that—

$$\underset{\cdot}{Z}_a = \frac{\underset{\cdot}{Z}_{\mathrm{I}}\,\underset{\cdot}{Z}_{\mathrm{III}}}{\underset{\cdot}{Z}_{\mathrm{I}} + \underset{\cdot}{Z}_{\mathrm{II}} + \underset{\cdot}{Z}_{\mathrm{III}}}$$

$$\underset{\cdot}{Z}_b = \frac{\underset{\cdot}{Z}_{\mathrm{I}}\,\underset{\cdot}{Z}_{\mathrm{II}}}{\underset{\cdot}{Z}_{\mathrm{I}} + \underset{\cdot}{Z}_{\mathrm{II}} + \underset{\cdot}{Z}_{\mathrm{III}}}$$

$$Z_c = \frac{\underset{\cdot}{Z}_{\mathrm{II}}\,\underset{\cdot}{Z}_{\mathrm{III}}}{\underset{\cdot}{Z}_{\mathrm{I}} + \underset{\cdot}{Z}_{\mathrm{II}} + \underset{\cdot}{Z}_{\mathrm{III}}}.$$

For the particular case in which each system is symmetrical—that is, when

$$Z_a = \underset{\cdot}{Z}_b = \underset{\cdot}{Z}_c \text{ and } Z_{\mathrm{I}} = \underset{\cdot}{Z}_{\mathrm{II}} = \underset{\cdot}{Z}_{\mathrm{III}}.$$

then $$\underset{\cdot}{Z}_a = \frac{1}{3}\,\underset{\cdot}{Z}_{\mathrm{I}},$$

that is $$R_a + jX_a = \tfrac{1}{3}R_1 + \tfrac{1}{3}jX_1$$

or $$R_1 = 3R_a : X_1 = 3X_a.$$

These results should be compared with those given in Chapter XVIII., p. 359.

An interesting special case is that in which the mesh arrangement of Fig. 460 consists of three equal capacities C_m.

For this case $\underset{\cdot}{Z}_{\mathrm{I}} = \underset{\cdot}{Z}_{\mathrm{II}} = \underset{\cdot}{Z}_{\mathrm{III}} = -j\frac{1}{\omega C_m}$,

where $$\omega = 2\pi \times \text{frequency}.$$

If C_s is the capacity per phase in the equivalent star connection,

then $$\underset{\cdot}{Z}_a = \underset{\cdot}{Z}_b = \underset{\cdot}{Z}_c = -j\frac{1}{\omega C_s}.$$

But it has been shown that

$$\underset{\cdot}{Z}_a = \tfrac{1}{3}\underset{\cdot}{Z}_{\mathrm{I}} = -j\,\frac{1}{3\omega C}.$$

Hence $$C_s = 3C\ .$$

See also p. 381.

SECTION VI

UNITS

CHAPTER XXIII

290. Fundamental Units.—There are three fundamental units of measurement—viz., a unit of length, a unit of mass or quantity of matter, and a unit of time. The magnitudes of all mechanical quantities can be expressed in terms of these units.

For scientific purposes the quantities chosen for the fundamental units of length, mass, and time are—

1. The **centimetre** as unit of length. The centimetre is very approximately one thousand-millionth part (*i.e.*, 10^{-9}) of a quadrant of the earth measured from the equator to the pole.

1 inch = 2·54 centimetres (very approximately).
1 metre = 100 centimetres = 39·35 inches (very approximately).
1 kilometre = 1000 metres = 0·62 mile.
8 kilometres = 5 miles (approximately).

2. The **gramme** as unit of mass. The gramme is very nearly the mass of 1 cubic centimetre of water at the temperature of its maximum density (4° C.).

1 gramme = 0·035 ounce.
1 kilogramme = 1000 grammes = 2·2 lbs. (very approximately).
1000 kilogrammes = 1 ton (approximately).

3. The **second** as unit of time. The second is $\frac{1}{86400}$ of the time of a mean solar day.

These fundamental units were chosen by a Congress of Electricians at Paris in 1881, and are almost universally adopted for all scientific purposes.

All units derived from these fundamental units of length, mass, and time are known as centimetre-gramme-second units, or more briefly as **c.g.s. units.**

291. Dimensions and Definitions of the Mechanical Units in Terms of the Fundamental Units.—The *velocity* of a body is its rate of displacement—that is, is measured by the ratio of $\frac{\text{length}}{\text{time}}$. In expressing the magnitude of a velocity it is therefore necessary to know the ratio of two quantities—viz., a length and a time—and the *dimensions* of a velocity are accordingly said to be (length) $(\text{time})^{-1}$, or, more briefly expressed—

$$LT^{-1}.$$

The *acceleration* of a body is the rate of change of its velocity—that is, is measured by $\frac{\text{change of velocity}}{\text{time}}$. The dimensions of an acceleration are thus said to be (velocity) $(\text{time})^{-1}$, or

$$LT^{-2}.$$

Force is measured by the product of an acceleration and a mass—that is, the dimensions of a force are thus—

$$LMT^{-2}.$$

The **dyne** or c.g.s. unit of force is that force which, acting on a mass of 1 gramme, produces a change of velocity of the mass of 1 cm. per second for every second during which the force acts.

The weight of 1 gramme—that is, the force of attraction of the earth on a mass of 1 gramme—is very nearly equal to 981 dynes.

Energy or *work* is measured by the product of a force into the distance through which the point of application of the force moves—that is, the dimensions.

The dimensions of energy are thus (force) (distance)—that is,

$$L^2MT^{-2}.$$

The **erg** or c.g.s. unit of work or energy is the work done when the point of application of a force of 1 dyne is moved through a distance of 1 centimetre.

This unit is rather small for many practical purposes, and a unit ten million times larger is also used, and is termed the **joule.**

$$1 \text{ joule} = 10^7 \text{ ergs}.$$

$$1 \text{ joule} = 0{\cdot}737 \text{ foot-lbs}.$$

Heat is a form of energy. In order to measure heat directly the *change of temperature* is used.

The change of temperature is measured by the change of volume of some substance such as mercury which takes place when heat is supplied to that substance.

The **British thermal unit** (**B. Th. U.**) is the heat necessary to raise the temperature of one pound of water 1 degree

Fahrenheit. The **calorie** is the heat necessary to raise the temperature of 1 gramme of water 1 degree Centigrade (actually from 4° C. to 5° C.).

1 calorie $=4 \cdot 2$ joules (very approximately); 1 joule $=0 \cdot 24$ calorie.

1 B. Th. U. $=252$ calories.

Power is the rate of doing work, and is therefore measured by the ratio $\frac{\text{work}}{\text{time}}$—that is, the dimensions of power are—

$$L^2MT^{-3}.$$

The **watt** is the c.g.s. unit of power. Work is being done at the rate of 1 watt when 1 joule of energy is expended in 1 second.

1 kilowatt $=1000$ watts; 1 horse-power $=746$ watts.

1 kilowatt-hour $=3{,}412$ B. Th. U.

292. Electrical and Magnetic Units.—It was stated in § 7 that when two concentrated electric charges or quantities of electricity q and q^1 are placed at a distance d cms. apart in a medium of which the specific inductive capacity is ϵ, each is acted on by a force F, such that

$$F=\frac{qq^1}{\epsilon d^2} \text{ dynes.}$$

If the two charges are of equal magnitude—that is, $q=q^1$,

then $$F=\frac{q^2}{\epsilon d^2},$$

or $$q^2=\text{force} \times \epsilon \times (\text{distance})^2.$$

Hence the dimensions of the square of an electric charge are ϵML^3T^{-2}, or the dimensions of a **quantity** of electricity are—

$$L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}\epsilon^{\frac{1}{2}}.$$

It appears, therefore, that the dimensions of a charge or quantity of electricity are not known until the dimensions of the specific inductive capacity ϵ are known. At present the dimensions of ϵ are not known (see § 55 and p. 476).

If it be assumed that ϵ is a pure number—that is, of no dimension in length, mass, or time—then the dimensions of an electric charge or quantity will be—

$$L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}.$$

This assumption that ϵ is a pure number is the basis of the electrostatic system of units (see § 293).

From the dimensions of an electric quantity the dimensions of other electrical conceptions may be deduced as follows:

Electric Current.—An electric current through a given area is the rate at which electric quantity passes across that area. The dimensions of an electric current are therefore—

$$\frac{\text{quantity}}{\text{time}} \text{ or } L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}\epsilon^{\frac{1}{2}}.$$

Intensity of Electric Force.—The intensity of the electric force at a distance d from a concentrated electric charge q is $\frac{q}{\epsilon d^2}$. The dimensions of the intensity of electric force are therefore—

$$\frac{\epsilon^{\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}}{\epsilon L^2}$$

or $$L^{-\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\epsilon^{-\frac{1}{2}}.$$

Electric Potential.—The electric potential between two points is measured by the work done in moving an electric charge from one point to the other—that is, the following relationship holds:

$$\text{potential} \times \text{quantity} = \text{work}.$$

The dimensions of electric potential are therefore—

$$\frac{\text{work}}{\text{quantity}};$$

that is, $$L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\epsilon^{-\frac{1}{2}}.$$

Electric Resistance is measured by the ratio $\frac{\text{potential}}{\text{current}}$, and the dimensions are therefore—

$$L^{-1}T\epsilon^{-1}.$$

Electric Capacity is measured by the ratio $\frac{\text{quantity}}{\text{potential difference}}$, and the dimensions are therefore—

$$L\epsilon.$$

Magnetic Flux.—The rate of change of magnetic flux through a circuit gives the e. m. f. induced in the circuit, or $\frac{\text{flux}}{\text{time}} =$ e. m. f. The dimensions of magnetic flux are therefore—

$$L^{\frac{1}{2}}M^{\frac{1}{2}}\epsilon^{-\frac{1}{2}}.$$

Magnetic Induction is the magnetic flux density or $\frac{\text{flux}}{\text{area}}$. The dimensions of magnetic induction are therefore—

$$L^{-\frac{3}{2}}M^{\frac{1}{2}}\epsilon^{-\frac{1}{2}}.$$

Intensity of Magnetic Force is the ratio of flux density to the permeability μ, and the dimensions of magnetic force are therefore—

$$L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-2}\epsilon^{\frac{1}{2}}.$$

Inductance.—The inductance multiplied by the rate of change of current in a circuit gives the induced e. m. f. or

$$\text{inductance}\times\frac{\text{current change}}{\text{time}}=\text{e. m. f.}$$

The dimensions of an inductance are therefore—

$$L^{-1}T^{2}\epsilon^{-1}.$$

All the above results have been deduced from the consideration of the force between two electric charges.

It is possible, however, to arrive at the dimensions of the electric and magnetic quantities by reference to the force between two magnetic poles. In § 85 it was stated that if two magnetic poles of strength m, m^1 be placed at a distance d cms. apart in a medium of magnetic permeability μ, the force F which will act on each charge is given by the expression—

$$F=\frac{mm^1}{\mu d^2}\text{ dynes.}$$

If

$$m=m^1:F=\frac{m^2}{\mu d^2}.$$

The dimensions of a **magnetic pole** or **quantity of magnetism** are therefore—

$$L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}.$$

The **magnetic flux** to which a magnetic pole of strength m gives rise is equal $4\pi m$, and hence the dimensions of magnetic flux are the same as those of magnetic quantity.

That is,

$$L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}.$$

From these results it is possible to deduce the dimensions of all the electrical and magnetic quantities, and these are summarised in the table on p. 476.

The dimensions of any electric or magnetic quantity must be the same, whether derived from a consideration of the force developed between two electric charges or of the force developed between two quantities of magnetism.

Thus the dimensions of a capacity derived from these two starting-points are respectively—

$$L\epsilon \text{ and } L^{-1}T^{2}\mu^{-1}.$$

Hence it follows that the dimensions of the product $\epsilon\mu$ are $L^{-2}T^{2}$, or the dimensions of $\frac{1}{\epsilon\mu}$ are $\frac{L^2}{T^2}$—that is, the square

Quantity.	*Dimensions Deduced by reference to Coulomb's Law for the Mutual Force Between Two Electric Quantities.*	*Dimensions Deduced by reference to Coulomb's Law for the Mutual Force Between Two Magnetic Poles.*
Electric quantity	$L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}\epsilon^{\frac{1}{2}}$	$L^{\frac{1}{2}}M^{\frac{1}{2}}\mu^{-\frac{1}{2}}$
Electric current	$L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}\epsilon^{\frac{1}{2}}$	$L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}$
Intensity of electric force ..	$L^{-\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\epsilon^{-\frac{1}{2}}$	$L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-2}\mu^{\frac{1}{2}}$
Electric potential or e. m. f.	$L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\epsilon^{-\frac{1}{2}}$	$L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}\mu^{\frac{1}{2}}$
Electric resistance	$L^{-1}T\epsilon^{-1}$	$LT^{-1}\mu$
Electric capacity	$L\epsilon$	$L^{-1}T^{2}\mu^{-1}$
Magnetic quantity or magnetic pole	$L\ M^{\frac{1}{2}}\epsilon^{-\frac{1}{2}}$	$L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}$
Magnetic flux	$L^{\frac{1}{2}}M^{\frac{1}{2}}\epsilon^{-\frac{1}{2}}$	$L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}$
Magnetic induction.. ..	$L^{-\frac{3}{2}}M^{\frac{1}{2}}\epsilon^{-\frac{1}{2}}$	$L^{-\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}$
Intensity of magnetic force	$L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-2}\epsilon^{\frac{1}{2}}$	$L^{-\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}$
Inductance	$L^{-1}T^{2}\epsilon^{-1}$	$L\mu$

of a velocity. The important result is thus arrived at that $\frac{1}{\sqrt{\epsilon\mu}}$ is a velocity.

If this velocity is v, then $\epsilon\mu v^2=1$.

The dimensions of the product of the specific inductive capacity and the permeability are thus known, although the dimensions of each quantity separately are not known.

In § 55 it was seen that $\frac{1}{\epsilon}$ was analogous to elasticity in elastic materials. It is also known that the velocity of wave motion in an elastic medium is given by the ratio $\sqrt{\frac{\text{elasticity}}{\text{density}}}$, and it is further known that the velocity v defined by the relationship $\epsilon\mu v^2=1$ is the velocity of electro-magnetic waves in ether. It follows, therefore, that if $\frac{1}{\epsilon}$ is a measure of the *elasticity* of the ether, μ is a measure of the *density*.

293. Electrostatic C.G.S. System of Units and Electromagnetic C.G.S. System of Units.—If for a vacuum it be assumed that the value of $\epsilon=1$, and the units of length, mass, and time are the c.g.s. units, then the electrical and magnetic quantities expressed in terms of these units are said to be expressed in **electrostatic c.g.s. units.**

Thus the electrostatic c.g.s. unit of electric charge or quantity of electricity is that quantity which, when concentrated at a point distant 1 centimetre from an equal and concentrated quantity in a vacuum (or air) is repelled by a force of 1 dyne (see § 8).

Similarly, if for a vacuum it be assumed that the value of $\mu=1$, and the units of length, mass, and time are the c.g.s. units, then the electrical and magnetic quantities expressed in terms of these units are said to be expressed in **electromagnetic c.g.s. units.**

Thus the electromagnetic c.g.s. unit of magnetism is that quantity which, when concentrated at a point distant 1 centimetre from an equal and concentrated quantity in a vacuum (or air), is repelled by a force of 1 dyne (see § 85).

The electromagnetic c.g.s. unit of electric current is that current which, when flowing in a circular conductor of 1 centimetre radius, exerts a force of 2π dynes on a unit magnetic pole placed at the centre of the circle (see p. 255).

The electromagnetic and electrostatic c.g.s. units are said to be **absolute** units. For most practical purposes, however, it is most convenient to use multiples of the electromagnetic and electrostatic c.g.s. units.

Quantity.	*Practical Unit.*	*No. of Electro-magnetic C.G.S. Units in the Practical Unit.*	*No. of Electro-static C.G.S. Units in the Practical Unit.*
Electric quantity	coulomb	10^{-1}	3×10^{9}
Electric current	ampère	10^{-1}	3×10^{9}
Electric capacity	farad and microfarad	10^{-9} in one farad 10^{-15} in one microfarad	9×10^{11} in one farad 9×10^{5} in one microfarad
Electric potential Difference	volt	10^{8}	$\frac{1}{3\times10^{2}}$
Electric resistance . .	ohm	10^{9}	$\frac{1}{9\times10^{11}}$
Inductance	henry	10^{9}	$\frac{1}{9\times10^{11}}$
Energy	joule	10^{7}	10^{7}
Power	watt	10^{7}	10^{7}

294. Experimental Determination of the Relationship between the Electrostatic and the Electromagnetic C.G.S. Systems of Units.—Suppose a spherical condenser (Fig. 462) is obtained of which the outer shell is formed of two accurately fitting hemispheres. Suppose the dimensions of the condenser are very accurately known. If the inner radius of the outer shell is b cms. and the radius of the inner sphere is a cms., then the capacity of this condenser in electrostatic c.g.s. units is—

$$\frac{ba}{b-a}.$$

Now let the condenser be charged to a potential of 1 electromagnetic c.g.s. unit—that is, 10^{-8} volts (§§ 197, 298), and let the quantity be measured by discharge through a ballistic galvanometer of which the constant has been calculated (see § 213). It will then be found that the capacity measured in electromagnetic c.g.s. units by the ballistic galvanometer will be $\frac{1}{(3\times10^{10})^2}$ times the capacity calculated in electrostatic c.g.s. units from the dimensions of the condenser—in other words, the number $(3\times10^{10})^2$ is the ratio of the magnitude of the electromagnetic unit of electrical capacity to the magnitude of the electrostatic unit of electrical capacity.

By §§ 292 and 293 it follows that the dimensions of the ratio $\frac{\text{capacity in electromagnetic c.g.s. units}}{\text{capacity in electrostatic c.g.s. units}}$ are those of $\frac{1}{(\text{velocity})^2}$, and hence the number 3×10^{10} represents a velocity. It is known from other experiments that the number 3×10^{10} expresses very approximately the velocity in centimetres per second of light and electromagnetic waves in a vacuum.

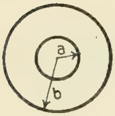


Fig. 462.

295. Standards of Electrical Measurements.—The fundamental units of length (the centimetre) and mass (the gramme) are recorded in concrete standards kept in Paris.

In the case of the electrical quantities, the only units which can be recorded in concrete standards are the units of resistance, capacity, and electromotive force. Of these standards that of resistance has been considered the most trustworthy, and special efforts have been concentrated on the measurement and production of a standard resistance of which the value is 1 ohm or 10^9 electromagnetic c.g.s. units (see § 141).

From the result of a number of expert determinations the conclusion was reached that *an ohm is represented by the resistance offered to an unvarying electric current by a column of mercury at the temperature of melting ice,* 14·4521 *grammes in mass, of a constant cross-sectional area, and of a length* 106·3 *centimetres.*

This is the legal definition of the ohm, and was specified in an Order in Council of 1894.

The data given for the column of mercury are very nearly the same as if the cross-section was given as 1 square millimetre and the length 106·3 centimetres at the temperature of 0° C.

296. Lorenz's Method for measuring Resistance in Electromagnetic C.G.S. Units.—Suppose in Fig. 463 A represents a metal disc rotated uniformly about its axis, the axis of rotation being placed at right angles to the magnetic meridian.

Let B be a concentric circular coil in which a current *C* electromagnetic c.g.s. units flows.

If *M* is the magnetic flux which passes through the plane of the disc when a current of unit strength flows in the coil, the flux through the plane of the disc when a current of C units flows in the coil will be—

CM c.g.s. unit lines.

If the disc is driven at a uniform rate of x revolutions per second, the e.m.f. induced between the centre of the disc and the periphery will be—

CMx electromagnetic c.g.s. units.

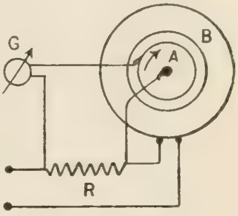


FIG. 463.

Now let the current through the coil *B* pass through a resistance R electromagnetic units. The potential difference between the ends of this resistance will be—

CR electromagnetic units.

If the centre and periphery of the rotating disc are electrically connected respectively to the ends of the resistance R through a galvanometer G it is possible, by making the connections the right way about and adjusting either the speed of the revolution x or the magnitude of the resistance R or the strength of the current C, to obtain a balance so that no deflection will be observed on the galvanometer G when the circuit of the galvanometer is completed. In this case—

$$\mathrm{CR} = \mathrm{CM}x,$$

or

$$\mathrm{R} = \mathrm{M}x.$$

The value of M can be calculated from the principles enunciated in §§ 181–187, and hence if the speed of revolution x is known the value of R is also known.

For the most recent application of Lorenz's method for the absolute measurement of resistance, reference should be made to F. E. Smith, National Physical Laboratory Report, 1914.

297. Measurement of Current in Electromagnetic C.G.S. Units.—The measurement of an electric current in absolute units may be made by means of the tangent galvanometer (p. 304), the constant of the galvanometer being determined by calculation from the dimensions of the coil, and the value of the horizontal component H being determined as explained in § 126.

A more convenient method, however, is that used by Lord Rayleigh. If two similar flat circular coils, of which the dimensions are accurately known, are arranged with their planes parallel and a small distance apart, and a current passed through the coils, there will be a force exerted between the coils. If one of the coils be movable the force may be measured and balanced by weights in a pan fixed to the movable coil. The force, however, may also be calculated from the dimensions of the coil and the current strength, according to the principles stated in Chapter XIV. Hence the magnitude of the current may be deduced from the weights necessary to balance the force between the coils and the dimensions of the coils.

If the current is passed through a silver voltameter (§ 161) in series with the coil, the weight of silver deposited will enable a determination of the electrochemical equivalent of silver to be made in terms of the electromagnetic c.g.s. mm. of current. From a number of careful experiments it has been deduced that *the electromagnetic c.g.s. unit of current is such that when flowing in a solution of silver nitrate it deposits silver at the rate of* 11·18 *milligrammes per second*; *or a current of* 1 *ampere is such that when flowing in a solution of silver nitrate it deposits silver at the rate of* 1·118 *milligrammes per second.*

This method of measuring a current by means of a silver voltameter gives a simple and accurate standard for which in any particular case it is only necessary to make a measurement of the *mass* of the deposited silver and the *time* for which the current flows.

298. Standard of Electromotive Force.—From the standards of resistance and current the measurement of an electromotive force in electromagnetic c.g.s. units may be made, since from Ohm's Law electromotive force = current × resistance.

It is found that the electromotive force of the Weston Normal Cell (p. 189) is 1·0183 volts at a temperature of 20° C.—that is, in electromagnetic c.g.s. units, the electromotive force of the Weston Normal Cell is $1{\cdot}0183 \times 10^8$ units.

APPENDIX

TABLE OF VALUES OF e^{-x}.

x	0	2	4	6	8
0·0	1·000	0·980	0·961	0·942	0·923
0·1	0·905	0·887	0·869	0·852	0·835
0·2	0·819	0·803	0·787	0·771	0·756
0·3	0·741	0·726	0·712	0·698	0·684
0·4	0·670	0·657	0·644	0·631	0·619
0·5	0·607	0·595	0·583	0·571	0·560
0·6	0·549	0·538	0·527	0·517	0·507
0·7	0·497	0·487	0·477	0·468	0·458
0·8	0·449	0·440	0·432	0·423	0·415
0·9	0·407	0·398	0·391	0·383	0·375
1·0	0·368	0·361	0·353	0·346	0·340
1·1	0·333	0·326	0·320	0·313	0·307
1·2	0·301	0·295	0·289	0·284	0·278
1·3	0·273	0·267	0·262	0·257	0·252
1·4	0·247	0·242	0·237	0·232	0·228
1·5	0·223	0·219	0·214	0·210	0·206
1·6	0·202	0·198	0·194	0·190	0·186
1·7	0·183	0·179	0·176	0·172	0·169
1·8	0·165	0·162	0·159	0·156	0·153
1·9	0·150	0·147	0·144	0·141	0·138
2·0	0·135	0·133	0·130	0·127	0·125

Values of e^{-x} intermediate to those tabulated above may be obtained by interpolation, thus:

$$e^{-1\cdot 03}=\tfrac{1}{2}(e^{-1\cdot 02}+e^{-1\cdot 04})=\tfrac{1}{2}(0\cdot 361+0\cdot 353)=0\cdot 357.$$

The following examples show how to obtain values of e^{-x} when x has a value greater than the highest value given in the table—*i.e.*, for x greater than 2·08.

$$e^{-3}=e^{-2}\times e^{-1}=0\cdot 135\times 0\cdot 368=0\cdot 050$$

$$e^{-2\cdot 66}=e^{-2}\times e^{-0\cdot 66}=0\cdot 135\times 0\cdot 517=0\cdot 070$$

$$e^{-6}=(e^{-2})^3=(0\cdot 135)^3=0\cdot 002.$$

It is also useful to note that

$$e^{-2\cdot 303}=0\cdot 1.$$

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